Title: Geometry for Technical Students
Date of first publication: 1904
Author: E. H. (Ernest Headly) Sprague
Date first posted: Aug. 29, 2018
Date last updated: Aug. 29, 2018
Faded Page eBook #20180889

This eBook was produced by: Ron McBeth, David T. Jones, Howard Ross & the online Distributed Proofreaders Canada team at http://www.pgdpcanada.net

Transcriber’s Note

Minor typographical corrections and presentational changes have been made without comment.

GEOMETRY FOR TECHNICAL STUDENTS

BEING

AN INTRODUCTION TO PURE AND APPLIED GEOMETRY AND THE MENSURATION OF SURFACES AND SOLIDS IN SIMPLE PROPOSITIONS

TO WHICH ARE ADDED

PROBLEMS IN PLANE GEOMETRY FOUND USEFUL IN DRAWING

BY

E. H. SPRAGUE, ASSOC.M.INST.C.E.

ASSISTANT IN THE ENGINEERING DEPARTMENT, UNIVERSITY COLLEGE, LONDON

LECTURER ON APPLIED MATHEMATICS AND MECHANICS AT THE WESTMINSTER TECHNICAL INSTITUTE

LATE PRINCIPAL OF THE IMPERIAL CHINESE RAILWAY COLLEGE SHAN-HAI-KUAN

LONDON

CROSBY LOCKWOOD AND SON

7, STATIONERS' HALL COURT, LUDGATE HILL

1904

PREFACE

If an apology be expected for putting forth yet another Text-book for the use of Students of Geometry, it must be based on the Author's experience—when training engineering students for the Chinese Government and in the positions he now holds—that a small volume upon the plan attempted to be carried out in the following pages is required by many students, more especially those who have to take up Geometry as part of their professional training as engineers.

The aim and scope of the work will be found explained in the Introduction (pp. 1, 2).

It is right that the Author should acknowledge here his great indebtedness to the German treatise of Schlomilch.

UNIVERSITY COLLEGE, LONDON.
October, 1903

LIST OF PROPOSITIONS

PROBLEMS IN PLANE GEOMETRY FOUND USEFUL IN DRAWING.

INTRODUCTION.

Etymologically the word *geometry* signifies the measurement of the earth.

Geometry had its origin in the practical needs of the Egyptians, but Thales of Miletus, about 600 B.C., first dealt with the subject in an *abstract* manner. Pythagoras and his school greatly added to the science, and Euclid, who taught at Alexandria about 300 B.C. collected together and arranged the labours of his predecessors in the famous “Elements,” which consists of thirteen books. This treatise has been in use up to the present time, with the exception of Books VII., VIII., IX., and X., which treat of Greek arithmetic and incommensurable magnitudes, and Book XIII., which treats of the regular solids.

The first six Books contain 164 propositions, and the XIth and XIIth Books 58 propositions. The greater number of these are merely links in the chain of reasoning by which the more important results are deduced. By starting with the theory of parallel lines, continental mathematicians have shown that a large number of the less important propositions can be omitted, and the same results obtained, without affecting the precision of the method; and this is the course adopted in the present work.

Moreover, by introducing early the idea of ratio, many of Euclid’s proofs are materially simplified, so that although, in the words of Euclid to Ptolemy, there may be no royal road to geometry, it is believed that the student may find in the following pages an easier guide to his requirements than our ordinary text-books.

It will be noted that the definitions of terms are distributed throughout the work as required for the elucidation of successive Propositions, and that in the case of many Propositions corollaries are stated, and examples appended, where deemed advisable.

Symbols.—The following symbols are commonly used for the sake of abbreviation:—

$\therefore$	meaning	*therefore.*	$\angle$	meaning	*angle.*
$\because$	"	*because.*	$\triangle$	"	*triangle.*
$=$	"	*equal to.*	$\square$	"	*parallelogram.*
$\parallel$	"	*parallel to.*	$\bigcirc$	"	*circle.*
$+$	"	*addition.*	$-$	"	*subtraction.*

GEOMETRY FOR TECHNICAL STUDENTS.

Preliminary Definitions.

1. **Geometry.**—Geometry is the science which treats of the properties of space.

The object of geometry is, starting from facts whose truth is universally recognised, to deduce therefrom results, the truth of which, being less apparent, can only be established by a chain of connected reasoning.

2. **Axiom.**—Self-evident facts are in geometry called *Axioms*—for example, "The whole of a thing is greater than a part of it;" "If equals be added to equals the wholes are equal;" and so on. Such axioms must be admitted as fundamental truths, for no proof can make them clearer.

3. **Proposition.**—A *Proposition* is the statement of something which it is required to do. Propositions are divided into Problems and Theorems.

4. **Problem.**—A *Problem* is a proposition which states that a certain thing is required to be done; *e.g.*, "To bisect a given angle."

5. **Theorem.**—A *Theorem* is a proposition which states that a certain assertion is to be proved true; *e.g.*, "Any two sides of a triangle are together greater than the third."

6. **Hypothesis.**—A Theorem consists of two parts; viz., the *Hypothesis*, or assumption; and the *Conclusion*, or that which follows from the reasoning based on the hypothesis. Thus,

If two sides of a triangle are equal (hypothesis)
The angles opposite to those sides are equal (conclusion).

7. **Converse.**—One Theorem is said to be the *Converse* of another, when the hypothesis and conclusion are interchanged. Thus, the converse of the above theorem would be

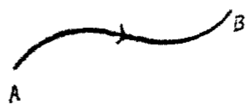
If two angles of a triangle are equal (hypothesis)
The sides opposite to those angles are equal (conclusion).

8. **Corollary.**—*Corollary* is a deduction which follows easily from a proposition already established.

The Elements of Geometrical Form.

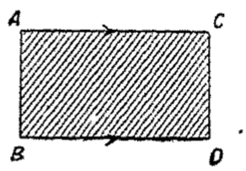
9. **Point.**—A *Point* is the smallest magnitude that can be imagined. It has *no dimensions*, that is, it has no measurement in any direction.

If a point be represented by a dot, this must only be regarded as a picture to show that the point has a certain position roughly indicated by the dot.



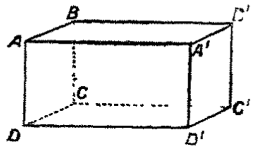
10. **Line.**—When a point moves it generates a *Line*. A line has *one dimension*—length; thus, *AB* is a line generated by a point which starts from *A*, and moves to *B*, along the path indicated by the line.

11. **Surface.**—In general, when a straight line moves, it generates a *Surface*. A surface has *two dimensions* at right angles to each other—length and breadth; thus, if the line *AB* move to *CD* along the path indicated, it will generate the surface *ABCD*



12. **Solid.**—In general, when a surface moves it generates *a solid*. A solid has *three dimensions* at right angles to one another—length, breadth, and thickness; thus, if a surface *ABCD* move to *A′B′C′D′*, it will generate the solid shown.

When a solid moves it generates another solid; consequently geometrical forms may be divided into *points*, *lines*, *surfaces*, and *solids*.



The Line.

13. **Straight Line**—14. **Curved Line.**—Lines are either *straight* or *curved*. A straight line is a line which is generated by a point moving always in the same direction, and is therefore the shortest distance between its extreme points. When a point continually changes its direction of motion, it generates a *curved* line.

A line may be affected in four ways; it may have

1. *Sense*; that is, it may be generated by the movement of a point from one extremity to the other, or *vice versâ*, as, from A to B in the one sense, or from B to A in the other.

2. *Direction*; that is, it has a definite inclination relatively to some fixed standard.

3. *Position*; that is, it has a definite place.

4. *Magnitude*; that is, it has a certain length.

The word *line* will, when used alone, signify a straight line.

Two Lines.

15. **Angle—Right Angle—Perpendicular—Acute Angle—Obtuse Angle—Reflex Angle.**—An *Angle* is the inclination of two lines to one another. Angles may be measured by the rotation of one of these lines relatively to the other. One-fourth of a complete revolution is called a right angle, and the lines are then said to be *perpendicular* to one another.

An *acute* angle is an angle less than a right angle.

An *obtuse* angle is an angle greater than one and less than two right angles.

An angle greater than two right angles is called a *reflex*, or *convex*, angle.

In practice, it is found convenient to measure angles by dividing a complete revolution into 360 equal parts, or *degrees*; these are sub-divided into 60 equal parts, or *minutes*; and these again into 60 equal parts, or *seconds*. Thus $35°4'22''$ denotes an angle of 35 degrees 4 minutes 22 seconds. A right angle is, of course, 90°.

16. **Complement.**—When two angles together make a right angle, either of them is said to be the *complement* of the other.

17. **Supplement.**—When two angles together make two right angles, either of them is said to be the *supplement* of the other.

18. **Parallel.**—When the angle between two lines is zero, the lines are *parallel*.

PROPOSITION 1.

When two lines cut one another, the opposite angles are equal.

Let AB, CD cut one another in E.

Now, if AB be supposed to rotate about E, until it coincide with CD, the parts AE and EB, since they turn together, must move through equal angles.

Consequently the angles AEC, BED, generated by rotation contraclockwise, must be equal.

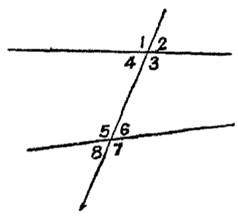
So also must the angles AED, BEC, which are generated when the coincidence is effected by clockwise rotation.

Three Lines.

19.—When a line intersects two other lines, it makes with them eight angles, 1, 2, 3, 4, 5, 6, 7, 8.

Of these—

1 and 5, 2 and 6, 3 and 7, 4 and 8 are called *corresponding* ∠s.



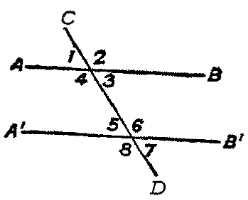
4 and 6, 3 and 5 are called *alternate* ∠s.

1, 2, 7 and 8 are called *exterior* ∠s.

3, 4, 5 and 6 are called *interior* ∠s.

PROPOSITION 2.

If one line meet two parallel lines (i.) the corresponding angles are equal; (ii.) the alternate angles are equal; (iii.) the exterior angles, and (iv.) the interior angles, on the same side of the line, are together equal to two right angles.



Let AB, $A'B'$ be any two parallel lines, and let CD meet them. Then since CD is equally inclined to both of them—

$\angle 1 = \angle 5; \angle 2 = \angle 6; \angle 3 = \angle 7; \angle 4 = \angle 8$ (by def. 19).

I.e., *the corresponding* ∠s *are equal.* (i.)

Again, since $\angle 4 = \angle 8, and \angle 6 = \angle 8$ (Prop. 1)

$$\therefore \angle 4 = \angle 6$$

Similarly, it may be proved that $\angle 3 = \angle 5$.

I.e., *the alternate* ∠s *are equal.* (ii.)

$$\begin{array}{rrcl} \text{Again, since} & \angle 1 + \angle 4 & = & 2 \text{ rt. } \angle\text{s} \\ \text{and} & \angle 4 & = & \angle 8 \text{ (by (i.))} \\ \therefore & \angle 1 + \angle 8 & = & 2 \text{ rt. } \angle\text{s} \end{array}$$

Similarly, it may be proved that $\angle 2 + \angle 7 = 2$ rt. ∠s.

I.e., *the exterior* $\angle$s *on the same side of the line are together equal to* 2 rt. $\angle$s. (iii.)

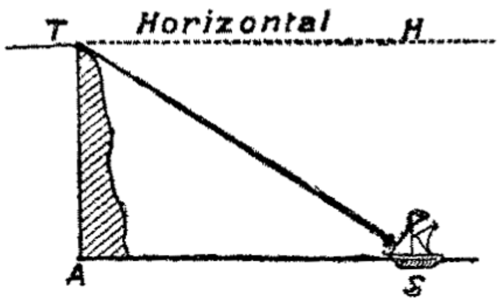
$$\begin{array}{rrcl} \text{Finally, since} & \angle 3 + \angle 4 & = & 2 \text{ rt. } \angle\text{s} \\ \text{and} & \angle 4 & = & \angle 6 \text{ (by (ii.))} \\ \therefore & \angle 3 + \angle 6 & = & 2 \text{ rt. } \angle\text{s.} \end{array}$$

Similarly, it may be proved that $\angle 4 + \angle 5 = 2$ rt. $\angle$s.

I.e., *the interior*$\angle$s *on the same side of the line are together equal to* 2 rt. $\angle$s. (iv.)

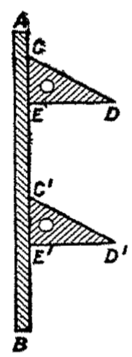
Corollary.—The converse of this proposition is obviously true also, viz., that if any of the statements (i.), (ii.), (iii.), or (iv.) be true, the lines AB, $A'B'$ must be parallel; for if they be not parallel, those statements cannot be true.

Example 1.



(i.) In calculating the horizontal distance AS of a ship S from the point A, vertically beneath the point of observation T, *the angle of depression* HTS is observed through which it is necessary to depress the instrument in order to sight on S. Then, since TH and AS are parallel, $\angle AST = \angle STH$ (by Prop. 2 (ii.)), and this, with the measurement of AT, enables the triangle to be drawn, or calculated. (See Prop. 5.)

Example 2.



The fact that when corresponding $\angle$s are equal the lines are parallel, is made use of by draughtsmen in drawing parallel lines; a straight-edge AB and set-square are employed. A line being drawn at ED, the square CED may be moved along the straight-edge to any position $C'E'D'$, and a line $E'D'$ drawn. This line will be parallel to ED, for the corresponding $\angle$s CED, $C'E'D'$ are equal. (By Prop. 2 (i.).)

20. **Triangle.**—When no two of three given lines are parallel they intersect in pairs, and form a *Triangle*, whose properties we next proceed to investigate.

Triangles.

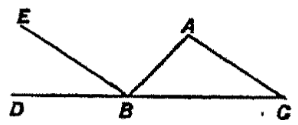
21. **Equilateral triangle.**—A triangle is *equilateral* when its sides are of equal length.

22. **Isosceles triangle**.—A triangle is *isosceles* when any two of its sides are of equal length.

PROPOSITION 3.

In any triangle, an exterior angle is equal to the two interior opposite angles.

Let ABC be any $\triangle$, and let any side CB be produced to D. Then if BE be a line parallel to CA, BE and CA are equally inclined to CD.

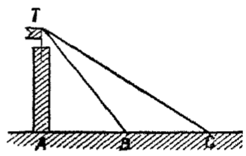


$$\therefore \angle DBE = \angle BCA \quad \text{(Prop. 2 (i.).)}$$
$$also \angle EBA = \angle BAC \quad \text{(Prop. 2 (ii.).)}$$
$$\therefore \text{the whole } \angle DBA = \angle BCA + \angle BAC.$$

COROLLARY.—An exterior angle of a triangle is greater than either of the interior opposite angles.

Example.

To calculate the height AT of an inaccessible object, on level ground, a base-line BC is measured, and the angles ABT, BCT are observed.



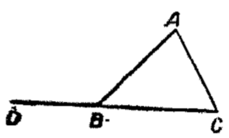
In order to calculate AT it is necessary to know the angle BTC. But by Prop. 3 $\angle ABT = \angle BTC + \angle BCT$

$\therefore \angle BTC = \angle ABT - \angle BCT$.

PROPOSITION 4.

The interior angles of any triangle are together equal to two right angles.

Let ABC be any triangle, and let any side CB be produced to D, then $\angle DBA = \angle BCA + \angle BAC$. (Prop. 3.)



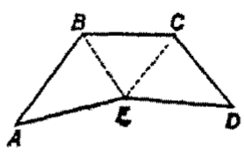
Add to each the $\angle ABC$, then $\angle DBA + \angle ABC = \angle BCA + \angle BAC + \angle ABC$

but $\angle DBA + \angle ABC = 2$ rt. $\angle$s

$\therefore \angle$ s BCA, BAC, $ABC = 2$ rt. $\angle$s.

Cor. 1.—Any two angles of a triangle are together less than two right angles.

Cor. 2.—All the interior angles of any polygon = twice as many right angles as the polygon has sides, less four right angles.



For since any polygon $ABCDE$ of n sides may be divided into two less $\triangle$s than there are sides, that is $n - 2\triangle$s, and since each $\triangle$ contains two rt. $\angle$s, the sum of all the $\angle$s must be $2n - 4$ rt. $\angle$s.

Example 1.

If the interior angles of a quadrilateral are 60°, 125°and 160°, find the remaining angle.

Answer 15°.

Example 2.

The interior angles of a six-sided polygon were observed to be 80°, 160°, 125°, 82°, 150°and 122°. What was the total error in the observations?

Answer 1°.

Conditions which determine a Triangle.

23. A triangle is said to be *determined*, when any other triangle constructed from the same data is *congruent* with it, that is to say, identical in every respect with it.

A triangle cannot be determined unless three parts at least are given. Hence there are four cases to consider:—

(i) When three angles are given.

(ii) When two angles and a side are given.

(iii) When one angle and two sides are given.

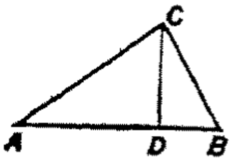
(iv) When three sides are given.

In the first case it is evident that any number of triangles can be drawn, having the angles in each respectively equal, and therefore, in this case, the triangle is not determined.

PROPOSITION 5.

A triangle is determined when two angles are given, and a side which is known to be either opposite or adjacent to these angles.

Suppose any side AB to be given, which is adjacent to the given angles A and B.



The directions of AC and BC are determined, because the angles BAC, ABC are known. Hence the point C, where AC and BC intersect, is determined.

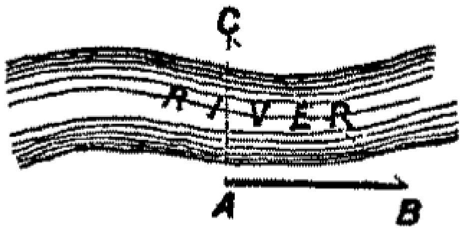
Again, if the $\angle$s A and C are given and the side AB opposite to one of these $\angle$s; then since $\angle B$ can be found (Prop. 4), the $\angle$s A and B are known, and this case becomes the same as the last, and the $\triangle$ is therefore determined.

Cor. 1.—Two triangles are congruent, when two angles in the one are equal to two angles in the other, and a side in each, either opposite or adjacent to one of the equal angles, are equal. For these triangles, being determined by the same data, must be identical.

Cor. 2.—If any two angles CAB, CBA in a triangle are equal, it follows, on supposing the angle at C bisected by CD, that the sides CB, CA opposite the equal angles are equal. For in this case the angles CAD, CBD have two angles and a side in each equal. Therefore by Cor. 1 they are congruent, and $AC = BC$.

Cor. 3.—If a triangle be equiangular, it is also equilateral. This follows immediately from Cor. 2.

Example.



Suppose it required to measure the width between A and C on opposite banks of a river. Measure a base-line AB on one side of the river, and at its extremities measure the angles ABC, CAB. Then the width AC is determined either by drawing to scale or by calculation (Prop. 5).

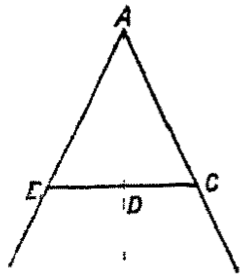
PROPOSITION 6.

A triangle is determined when one angle and two sides are given; except when the given angle lies opposite the smaller of the given sides;

in which case there are two triangles having supplementary angles.

Case 1.—When the given angle is included by the given sides.

Let BAC be the given angle, and AB, AC the given sides.



Then since only one straight line can be drawn between B and C, the triangle is determined.

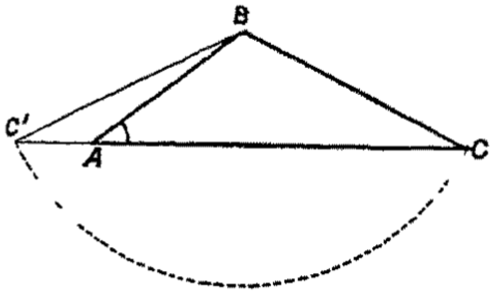
COR. 1.—If the given side BA = the given side CA, and AD be the line which by supposition bisects the angle BAC, then in the $\triangle$s ABD, ACD, the sides AB, AD in the one are = the sides AC, AD in the other, and the included $\angle$s are equal; the $\triangle$ s are therefore congruent (by the preceding), and $\angle ABC = \angle ACB$.

I.e., the angles at the base of an isosceles triangle are equal.

COR. 2.—An equilateral triangle is also equiangular. This follows immediately from the preceding.

Case 2.—When the given angle is not included by the given sides.

Let the sides AB, BC and the $\angle A$ be given; and first, let the greater side BC be opposite the given $\angle A$.

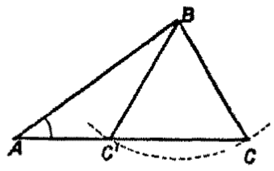


Then since the point C must lie on AC or AC produced, and must also be at a distance from B = BC, it must lie somewhere on the circle whose centre is B and radius BC, and must therefore be either at C or C' where the line AC cuts the circle. In the first case we have the $\triangle ABC$, and in the second case we have the $\triangle ABC'$. But of these, the $\triangle ABC$ alone contains the given $\angle A$. Therefore the $\triangle$ is determined.

Next, let BC opposite the given$\angle A$, be the smaller side.

In this case the point C' lies between A and C, and we get two $\triangle$s ABC, ABC', both having their sides AB, BC

and AB, BC' equal to the given sides, and containing the given $\angle A$.



In this case, therefore, there are two $\triangle$s satisfying the data, but the $\angle$s BCA, $BC'A$ are supplementary.

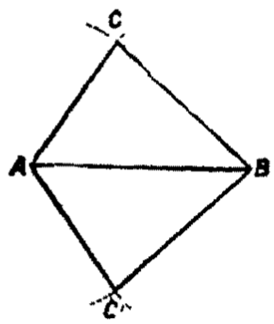
This case is known as the "ambiguous case."

COR. 3.—Triangles are congruent when one angle and two sides in each are equal, except when the given angle lies opposite the smaller of the given sides, in which case the triangles may be congruent or may have supplementary angles.

PROPOSITION 7.

A triangle is determined when the three sides are given.

Let AB be one of the sides.



Then, since the vertex of the $\triangle$ must lie on the circumference of a circle whose centre is A and radius AC, equal to one of the remaining sides, and likewise on the circumference of a circle whose centre is B and radius BC, equal to the third side, it must be either at C or C', the points where these circles cut one another. But, from the nature of the construction, AB is an axis of symmetry, and therefore the $\triangle$s ABC, ABC' are congruent.

COR. 1.—Triangles which have the three sides of the one equal respectively to the three sides of the other, are congruent.

Hence, triangles are congruent, if

(i) two angles and a corresponding side in each, are equal;

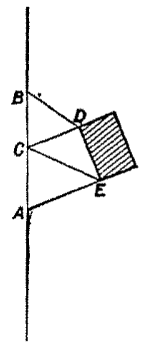
(ii) one angle and two sides in the one are equal to one angle and two sides in the other, except when the given angle is opposite the smaller side; in which case two triangles can be formed, one of which is congruent with the first, and the other not;

(iii) the three sides in the one are equal to the three sides in the other; that is to say, if in two triangles three elements in the one are known to be equal to the three corresponding elements in the other, the triangles are congruent in all cases, except

(i) when three angles are equal.

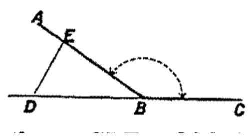
(ii) in the ambiguous case.

Practical applications of the congruence of triangles:—

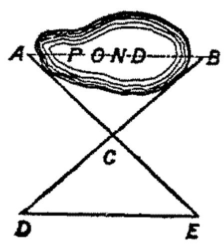


1. To determine the position of a point by offsets from a straight line.

Suppose AB to be a survey line, and it is required to fix the position of the corners of a building D, E with respect to it. Take any convenient points A, C, B in the line, and measure BD, CD, CE, AE. Then the $\triangle$s BCD, ACE are determined. (Prop. 7.)



2. Let it be required to determine an angle ABC with a chain only. Produce CB to any convenient length BD. Set off the same or any other convenient length BE along BA, and measure DE. Then the $\triangle DBE$ is determined, and therefore also the $\angle CBE$, which is supplementary to it. (Prop. 7.)



3. To measure an inaccessible distance by means of a chain only. (Prop. 6.)

Let AB be the inaccessible distance.

Take any convenient point C, and produce AC, BC until $CE = AC$ and $CD = BC$. Then the $\triangle$s ABC, CDE are congruent, and $\therefore DE = AB$.

Exercises.

1. To draw a straight line perpendicular to a given straight line from a given point in it.

2. To draw a straight line perpendicular to a given straight line from a given point without it.

3. Any two sides of a triangle are together greater than the third side.

4. To trisect a right angle.

5. To construct a triangle, having given the base, an angle at the base, and the sum of the sides.

6. The bisectors of the angles of a triangle are concurrent.

7. The perpendicular bisectors of the sides of a triangle are concurrent.

8. The medians of a triangle are concurrent, and the point of intersection is one-third of any median from the corresponding side.

9. The straight line which joins the points of bisection of the sides of a triangle is parallel to the base, and one-half of it.

10. If the adjacent sides of a quadrilateral are bisected and the points joined, the figure so formed is a parallelogram.

11. Given two straight lines and a given point between them. To draw through the given point a straight line which shall be bisected in that point.

12. Given two angles of a triangle and the perimeter; to construct the triangle.

13. Through a given point, to draw a straight line that shall make equal angles with two given straight lines.

Figures Consisting of Four Lines.

24. A *parallelogram* (▱) is a four-sided figure having its opposite sides parallel.

25. A *rhombus* is an equilateral parallelogram.

26. A *rectangle* is a right-angled parallelogram.

27. A *square* is an equilateral rectangle.

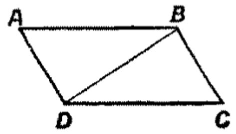
28. A *trapezium* is a four-sided figure having two of its sides parallel.

29. A *quadrilateral* is any plane four-sided figure.

30. A line which joins any two non-adjacent corners of a polygon is called a *diagonal*.

PROPOSITION 8.

A parallelogram is bisected by its diagonal.



Let $ABCD$ be any ▱, and BD a diagonal.

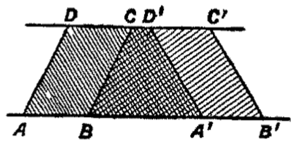
Then in the $\triangle$s ABD, BCD,

$\because$ $\angle ABD = \angle BDC$ and $\angle ADB = \angle DBC$ (Prop. 2), and the side BD is common, $\triangle ABD = \triangle BDC$. (Prop. 5, Cor. 1.)

PROPOSITION 9.

Parallelograms on equal bases and between the same parallels are equal in area.

Let $ABCD$, $A'B'C'D'$ be two ▱s on equal bases AB, $A'B'$ and between the same parallels DC', AB'.



Then if the trapezium $AA'D'D$ be moved parallel to AB', until A' coincides with B', it will coincide with the trapezium $BB'C'C$, since $AB = A'B'$.

Hence $AA'D'D$ and $BB'C'C$ are equal in area.

From each take away the trapezium $CD'A'B$.

Then the remaining ▱$ABCD =$ remaining ▱$A'B'C'D'$.

Cor. 1.—Parallelograms on equal bases and of equal altitude are equal in area.

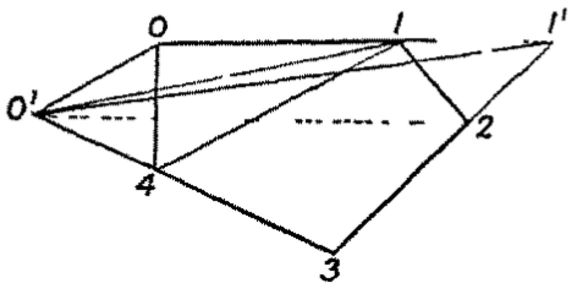
Cor. 2.—Triangles on equal bases and of equal altitude are equal in area.

Cor. 3.—A triangle has half the area of a parallelogram on the same base. (Prop. 8 and Cor. 2.)

Example.

To reduce a rectilinear figure to a triangle of equal area.

Let 012340 be any rectilinear figure.



Join 14 and through 0 draw 00′ parallel to 14 to meet the side 34 produced in 0′.

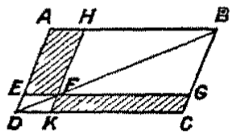
Then $\triangle 014 = \triangle 0'14$, and the quadrilateral $0'1230'$ is equal to the five-sided figure 012340. (Prop. 9, Cor. 2.)

Similarly, by joining $0'2$ and drawing $11'$ parallel $0'2$ to meet 32 produced in $1'$, the quadrilateral is reduced to the equal $\triangle 0'1'3$.

PROPOSITION 10.

Parallelograms about the diagonal of any parallelogram are equal.

Let AF, FC be ▱s about the diagonal of any ▱$ABCD$.



Then ∵ the $\triangle ABD = \triangle BCD$ (Prop. 7, Cor. 1)

$\triangle DEF = \triangle DKF$ (Prop. 7, Cor. 1)

and $\triangle FHB = \triangle FGB$ (Prop. 7, Cor. 1)

$\therefore \triangle ABD - \triangle DEF - \triangle FHB = \triangle BCD - \triangle DKF - \triangle FGB$

i.e. ▱AF = ▱FC.

Exercises.

1. When equal triangles stand on equal bases in one straight line and on the same side of it, they are of equal altitude, or lie between the same parallels.
2. To draw a triangle, the altitude and the base angles being given.

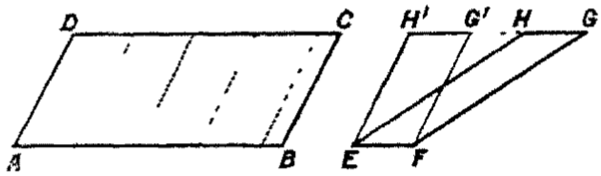
31. **Ratio.**—The *ratio* of one quantity to another is the fraction which expresses the numerical relation between their magnitudes. Thus $\frac{A}{B}$ is the ratio of A to B.

32. **Proportion.**—When two ratios are equal, the quantities which constitute the ratios are said to be in *proportion*.

Thus, if $\frac{A}{B} = \frac{C}{D}$, A, B, C, D are in *proportion*.

PROPOSITION 11.

Parallelograms of equal altitude are to one another as their bases.



Let $ABCD$, $EFGH$ be any two $\square$s of equal altitude.

If EH', FG' be drawn $\parallel$ AD or BC the $\square EFG'H' = \square EFGH$. (Prop. 9.)

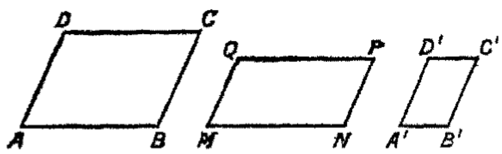
But the $\square ABCD$ may be divided into as many $\square$s equal to $EFG'H'$ and parts of it as the base AB contains the base EF.

$\therefore \frac{\square ABCD}{\square EFGH} = \frac{AB}{EF}$, *i.e.* the $\square$s are proportional to their bases.

COR.—Triangles of equal altitude are to one another as their bases.

PROPOSITION 12.

The areas of equiangular parallelograms are as the products of their sides.



Let $ABCD$, $A'B'C'D'$ be two equiangular $\square$s.

Construct a $\square MNPQ$ having the same angles, and with one side $MN = AB$, and the other $NP = B'C'$.

$$\text{Then } \frac{\square ABCD}{\square MNPQ} = \frac{BC}{NP} \text{ (Prop. 11.)}$$

$$\text{and } \frac{\square A'B'C'D'}{\square MNPQ} = \frac{A'B'}{MN} \text{ (Prop. 11.)}$$

$$\therefore \text{ by division, } \frac{\square ABCD}{\square A'B'C'D'} = \frac{MN \cdot BC}{A'B' \cdot NP} = \frac{AB \cdot BC}{A'B' \cdot B'C'}$$

COR.—The areas of triangles having an angle in each equal, are as the products of the sides about that angle.

Example 1.

$ABCD$, $A'B'C'D'$ are two equiangular ▱s, the area of the first being three times that of the second; if $AB = 2'$, $BC = 3'$, and $A'B' = 1'$, find $B'C'$.

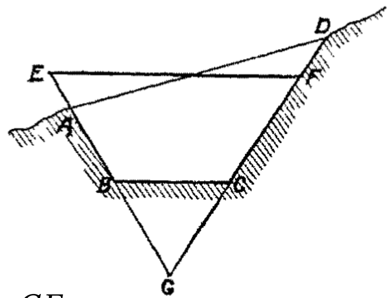
Answer $2'$.

Example 2.

In the preceding, if $AB = 4'$, $BC = 5'$, $A'B' = 3'$ and $B'C' = 4'$, and the area of $A'B'C'D'$ be 9 square feet, find the area of $ABCD$.

Answer 15 square feet.

Example 3.



Let it be required to find the length EB of the side of a level cutting $EBCF$ having the same area as another cutting $ABCD$, the ground surface of which is not level.

Produce AB, DC to meet in G.

$$\text{Then } \frac{\triangle EGF}{\triangle AGD} = \frac{EG \cdot GF}{AG \cdot GD}$$

But $\triangle EGF = \triangle AGD$ (by Hyp.)

$$\therefore \frac{EG \cdot GF}{AG \cdot GD} = 1 \text{ and } \therefore EG \cdot GF = AG \cdot GD.$$

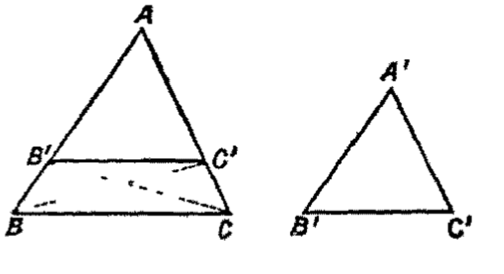
But when the side-slopes are equal, as is generally the case, $EG = GF$ and $\therefore$ $EG = \sqrt{AG \cdot GD}$ and $\therefore$ $EB = \sqrt{AG \cdot GD} - \overline{BG}$.

Similar Figures.

33. **Similar figures** are equiangular, and have their corresponding sides proportional.

PROPOSITION 13.

Equiangular triangles are also similar.



Let ABC, $A'B'C'$ be equiangular $\triangle$s, having the $\angle A = \angle A'$.

Apply the $\triangle A'B'C'$ to the $\triangle ABC$, so that A' falls on A, and $A'B'$ on AB; then will $A'C'$ fall on AC, because $\angle A' = \angle A$

and $\because$ $\angle A'B'C' = \angle ABC$ (by Hyp.)

$B'C'$ is parallel to BC. (Prop. 2.)

Join $B'C$, BC'.

$$\text{Then } \frac{\triangle ABC'}{\triangle AB'C} = \frac{AB \cdot AC'}{AC \cdot AB'} \quad \text{(Prop. 12, Cor.)}$$

$$\text{But } \triangle ABC' = \triangle AB'C' + \triangle BB'C'$$

$$\text{and } \triangle AB'C = \triangle AB'C' + \triangle B'C'C$$

$$\text{But } \triangle BB'C' = \triangle B'C'C. \quad \text{(Prop. 9, Cor. 2.)}$$

$$\therefore \triangle ABC' = \triangle AB'C$$

$$\text{Consequently } \frac{AB \cdot AC'}{AC \cdot AB'} = 1$$

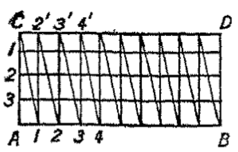
$$\textit{i.e.} \ \frac{AB}{AC} = \frac{AB'}{AC'} = \frac{A'B'}{A'C'}$$

$$\text{Similarly } \frac{BC}{BA} = \frac{B'C'}{B'A'}$$

$$\text{and } \frac{CA}{CB} = \frac{C'A'}{C'B'}$$

Example.

Diagonal Scale.



The diagonal scale enables us to subdivide a small distance very accurately. Thus, if AB be a line, which it is required to divide into 40 equal parts, say. Divide AB into 10 equal parts, and set up a perpendicular AC of any convenient length, and divide it into four parts; then if horizontal and vertical lines be drawn, and also diagonals $C1$, $22'$, $33'$, &c., each of these divisions will be subdivided into four equal parts (by Prop. 13), and the whole line therefore into 40 equal parts.

Example 1.

Draw a diagonal scale 6 inches long, to read 1/100ths of an inch.

Example 2.

A mechanical drawing is made in terms of a unit whose length is 1.25 inches. Draw a diagonal scale to give tenths and hundredths of the unit.

Example 3.

Draw a diagonal scale of 60 chains to an inch, to read chains.

Conditions which Determine the Similarity of Triangles.

It follows from the preceding propositions that two triangles are similar when—

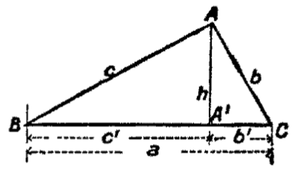
1. One angle in the one is equal to one angle in the other, and the ratio of the sides about these angles are equal;
2. One angle in the one is equal to one angle in the other, and the ratio of any two sides in each are equal, provided that the triangle is determined by the three parts considered;
3. Any two side-ratios in each are equal;
4. Any two angles in each are equal.

PROPOSITION 14.

In any right-angled triangle, the perpendicular on the hypothenuse from the opposite vertex makes triangles which are similar to the

whole triangle and to one another.

Let ABC be any rt.- $\angle$d $\triangle$, having a rt. $\angle$ at A, and let $AA' = h$ be the perpendicular on the hypothenuse BC.



Then, in the $\triangle$s ABC, ABA', the $\angle$s BAC, $BA'A$ are rt. $\angle$s, and the $\angle B$ is common, $\therefore$ the $\triangle$s are similar.

In the same way the $\triangle$s ABC, ACA' are similar.

$\therefore$ the three $\triangle$s ABC, ABA', ACA' are similar.

COR. 1.—Since the $\triangle$s ABA', ACA' are similar

$$\frac{c'}{h'} = \frac{h}{b'} \quad \text{(Prop. 13)} \qquad \therefore h^2 = b'c'$$

i.e. the square on the perpendicular is equal to the rectangle contained by the segments into which it divides the hypothenuse.

COR. 2.—Since the $\triangle$s ABA', ABC are similar

$$\frac{c}{a} = \frac{c'}{c} \quad \text{(Prop. 13)} \qquad \therefore c^2 = ac'$$

Similarly since the $\triangle$s ACA', ABC are similar

$$\frac{b}{a} = \frac{b'}{b} \quad \text{(Prop. 13)} \qquad \therefore b^2 = ab'$$

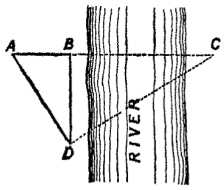
i.e. the square on one side of a rt. $\angle$d $\triangle$ is equal to the rectangle contained by the hypothenuse and the projection of that side on the hypothenuse.

COR. 3.—Hence

$$\begin{aligned} b^2 + c^2 &= ab' + ac' \quad \text{(Cor. 2.)} \\ &= a(b' + c') = a \times a = a^2 \end{aligned}$$

I**mportant.**—*I.e.* In any rt.- $\angle$d $\triangle$, the squares on the sides about the rt. $\angle$ are together equal to the square on the hypothenuse.

Example.



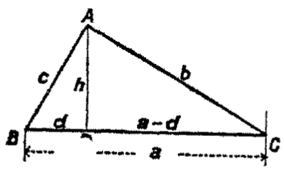
To measure the width of a river, &c., indirectly.

Measure a distance AB, and make $BD = 2AB$, say; BD being at rt. $\angle$s to AB. Sight C in line with AB, and make ADC a rt. $\angle$. Then $BD^2 = AB \cdot BC$

$$\therefore BC = \frac{BD^2}{AB} = \frac{4AB^2}{AB} = 4AB.$$

PROPOSITION 15.

In any triangle the square on one side is less than the sum of the squares on the other two sides, by twice the area of the rectangle contained by either of these and the projection on it of the other.



Let ABC be any $\triangle$, and let AD be perp. to BC, then BD is the projection of AB on BC.

Now $b^2 = h^2 + (a-d)^2$. (Prop. 14, Cor. 3).

$= h^2 + a^2 - 2ad + d^2$. (By Algebra.)

but $h^2 + d^2 = c^2$. (Prop. 14, Cor. 3.)

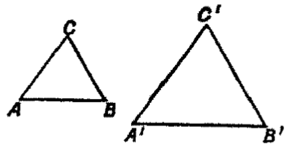
$\therefore b^2 = a^2 + c^2 - 2ad.$

If the angle B be obtuse, then the projection d will be on the side produced, and, being drawn in the opposite direction to what it was before, must be regarded as a negative quantity, so that the last term will then be added instead of subtracted. With this convention, Prop. 15 holds for any triangle.

PROPOSITION 16.

The areas of similar triangles are in the ratio of the squares of their corresponding sides.

Let ABC, $A'B'C'$ be any similar $\triangle$s.

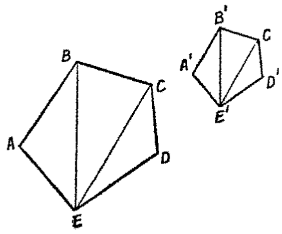


$$\text{Then } \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB \cdot AC}{A'B' \cdot A'C'} \quad \text{(Prop. 12, Cor.)}$$

$$\text{But } \frac{AC}{A'C'} = \frac{AB}{A'B'} \quad \text{(Prop. 13.)}$$

$$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB \cdot AB}{A'B \cdot A'B'} = \left(\frac{AB}{A'B'}\right)^2 = \left(\frac{AC}{A'C'}\right)^2 = \left(\frac{CB}{C'B'}\right)^2$$

PROPOSITION 17.



The areas of similar polygons are in the ratio of the squares of their corresponding sides.

Let $ABCDE$, $A'B'C'D'E'$ be similar polygons, which are made up of the similar $\triangle$s ABE, $A'B'E'$, &c.

$$\text{Then since } \frac{\triangle ABE}{\triangle A'B'E'} = \frac{AB^2}{A'B'^2}$$

$$\text{and } \frac{\triangle BCE}{\triangle B'C'E'} = \frac{BC^2}{B'C'^2} \ \&c. \quad \text{(Prop. 16)}$$

$$\text{and since } \frac{AB}{A'B'} = \frac{BC}{B'C'} = \&\text{c.} \quad \text{(Prop. 13.)}$$

$$\therefore \frac{\triangle ABE}{\triangle A'B'E'} = \frac{\triangle BCE}{\triangle B'C'E'} = \&\text{c.} = \left(\frac{AB}{A'B'}\right)^2$$

$$\therefore \frac{\triangle ABE + \triangle BCE + \&\text{c.}}{\triangle A'B'E' + \triangle B'C'E' + \&\text{c.}} = \left(\frac{AB}{A'B'}\right)^2 \quad \text{(by Algebra).}$$

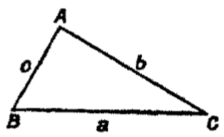
$$\therefore \frac{ABCDE}{A'B'C'D'E'} = \left(\frac{AB}{A'B'}\right)^2 = \left(\frac{BC}{B'C'}\right)^2 = \&\text{c.}$$

and similarly for any other polygons.

COR.—The areas of similar figures are in the ratio of the squares of their corresponding linear dimensions.

PROPOSITION 18.

The area of any figure described on the hypothenuse of a right-angled triangle is equal to the similar and similarly described figures on the sides about the right angle.



Let ABC be any rt.- ∠d △. Then if M_1 and M_2 be the areas of the figures on b and c, the sides about the rt. $\angle A$, and M the area of the similar figure on a, then

$$\frac{M_1}{M} = \frac{b^2}{a^2} \qquad \text{and} \qquad \frac{M_2}{M} = \frac{c^2}{a^2} \quad \text{(Prop. 17).}$$

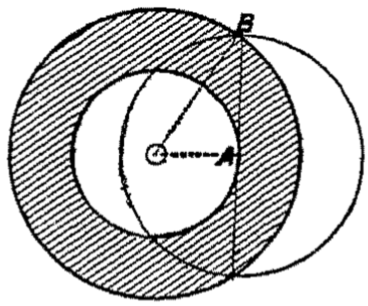
Hence, by addition,

$$\frac{M_1 + M_2}{M} = \frac{b^2 + c^2}{a^2} = \frac{a^2}{a^2} = 1 \quad \text{(Prop. 14, Cor. 3).}$$

$$\therefore M_1 + M_2 = M$$

Practical Example.

To substitute for a hollow round column a solid one of equal area.



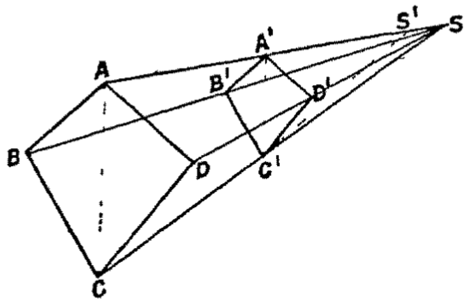
Let OA be the internal radius, and OB the external radius of the hollow column. Draw BA tangent to the inner circle.

Then a circle described with AB as radius will have the same area as the difference between the other two.

Proof.—For since OAB is a rt.- ∠△, by Prop. 18, the circles described with the sides OA, AB, as radii, will be equal in area to the circle described with the hypothenuse OB as radius.

PROPOSITION 19.

If any two similar figures are placed with their corresponding sides parallel, the lines joining corresponding points in the two figures are concurrent.



Let $ABCD$, $A'B'C'D'$ be two such figures, and let AA' meet BB' in S.

Then, $\because$ AB is par. to $A'B'$ (by Hyp.), the $\triangle$s SAB, $SA'B'$ are similar,

and $\therefore$ $\frac{SB}{SB'} = \frac{AB}{A'B'}$ (Prop. 13.)

Again, if possible, let CC' meet BB' in the point S' not coincident with S.

Then $\because$ BC is $\parallel$ to $B'C'$ (by Hyp.), the $\triangle$s $S'BC$, $S'B'C'$ are similar,

$$\text{and } \therefore \frac{S'B}{S'B'} = \frac{BC}{B'C'} \quad \text{(Prop. 13.)}$$

But since ABC, $A'B'C'$are similar $\triangle$ s (Hyp.)

$$\frac{AB}{BC} = \frac{A'B'}{B'C'} \quad \text{(Prop. 13.)}$$

$$\text{and } \therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} \quad \text{(By Algebra.)}$$

$$\therefore \frac{SB}{SB'} = \frac{S'B}{S'B'}$$

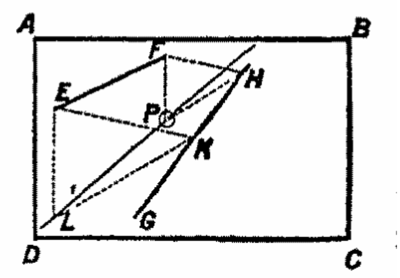
$$\text{Hence, } \frac{SB - SB'}{SB'} = \frac{S'B - S'B'}{S'B'} \quad \textit{i.e.,} \quad \frac{BB'}{SB'} = \frac{BB'}{S'B'}$$

$\therefore$ $SB' = S'B'$, and $\therefore$ S and S' must be coincident.

In the same way it may be shown that DD' must pass through S.

Practical Example.

To draw a line through a given point to pass through the inaccessible join of two given lines.



Let *ABCD* be a drawing-board, and let *EF* and *GH* be two lines upon it which intersect beyond the limits of the board.

It is required to draw through a given point *P* a straight line which shall pass through the intersection of the given lines produced. Draw any $\triangle PFH$, and draw $EK \parallel$ to FH. Through *E* and *K* draw lines *EL*, *KL* par. to *FP*, *PH* intersecting in *L*. Then *LP* produced will pass through the intersection of the given lines, as required (by Prop. 19).

EXERCISE.—Draw a line to pass through the inaccessible points which are given by two pairs of lines.

The Circle.

34. A *circle* is a figure contained by the path of a point which rotates about a fixed point or *centre*, at a constant distance from it, called the *radius*. The path of the point is the *circumference* of the circle, and any line through the centre is called a *diameter*.

35. A *chord* of a circle is the straight line joining any two points on its circumference.

36. A *tangent* is a line which touches a circle.

37. A *secant* is a line which cuts a circle.

38. A *sector* of a circle is the figure contained by an arc and the radii at its extremities.

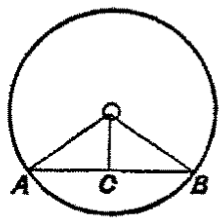
39. A *segment* of a circle is the figure contained by a chord and an arc of the circle.

40. *Concentric* circles are circles having a common centre.

41. An *arc* of a circle is part of its circumference.

PROPOSITION 20.

The straight line drawn from the centre of a circle to the middle point of a chord is perpendicular to the chord.



Let AB be any chord bisected in C. Join OA, OB.

Then in the $\triangle$s OAC, OBC, since the three sides in each are respectively equal, the $\angle OCA = \angle OCB$ (Prop. 7, Cor. 1), and therefore each of them is a right angle.

$\therefore$ OC is perpendicular to AB.

COR. 1.—Conversely, the straight line drawn from the point of bisection of a chord perpendicular to it, passes through the centre.

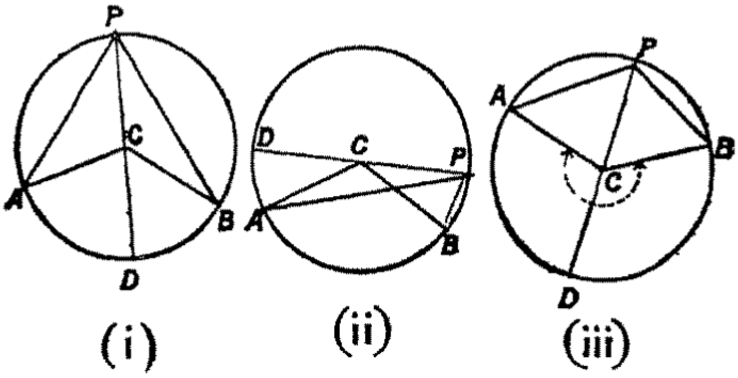
COR. 2.—The line from the centre of a circle perpendicular to a chord, bisects the chord.

COR. 3.—If the points A and B approach each other indefinitely, the chord becomes a tangent, and the line OC, which bisects AB, becomes the radius at the point of contact, and therefore a radius and the tangent at its extremity are at right angles.

PROPOSITION 21.

The angle at the centre of a circle is double the angle at the circumference standing on the same arc.

There are three cases to be considered, as in (i.), (ii.), (iii.), where the $\angle APB$ is the $\angle$ at the circumference, and ACB the $\angle$ at the centre standing on the same arc AB.



Then, in each case, $\angle ACD = \angle CAP + \angle CPA = 2\angle CPA$ (Prop. 3 and Prop. 7, Cor. 1),

and $\angle BCD = \angle CBP + \angle CPB = 2\angle CPB$ (Prop. 3 and Prop. 7, Cor. 1).

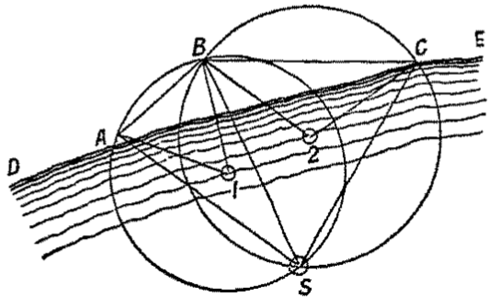
$\therefore$ in Case (i.) and Case (iii.) by Addition $\angle ACB = 2\angle APB$,

and in Case (ii.) by Subtraction $\angle ABC = 2\angle APB$.

COR. 1.—All angles at the circumference standing on the same arc are equal.

COR. 2.—When the angle at the centre is equal to two right angles, the angle at the circumference is one right angle, that is to say, *the angle in a semicircle is a right angle*.

Example 1.

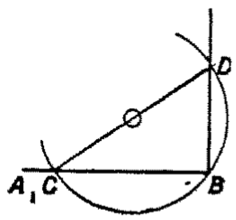


To determine the position of a ship at sea by observations on three known points ashore. (Three-point Problem.) Let A, B, C be the known points on shore, DE being the shore-line, and let S be the ship. The angles ASB, BSC are observed. If a ○ be described on AB containing the angle equal to the observed angle ASB, the point S must lie upon it (Cor. 1). Similarly, if a ○ be described on BC containing an angle equal to the observed angle BSC, S must lie upon it also. Consequently S must lie at their point of intersection. In order to construct these circles, double the observed angle to find the angle at the centre (Prop. 21), subtract this from two right angles, and halve the remainder. The result will be the angles at the base of the $\triangle sAB1$, $BC2$.

Example 2.

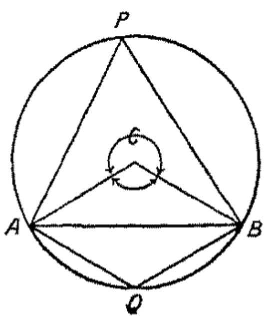
To draw a perpendicular at B to a given line AB at its extremity without producing it.

Let AB be the given line. Take a convenient point O, and with OB as radius describe a circle cutting AB in C. Join CO, producing it to cut the circle in D. Then BD is the perpendicular required, for the angle CBD, being the angle in a semi-○, is a right angle. (Cor. 2.)



PROPOSITION 22.

The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.



Let $APBQ$ be a quadrilateral in a circle.

Then, since $\angle APB = \frac{1}{2}$ concave angle ACD. (Prop. 21),

and $\angle AQB = \frac{1}{2}$ convex angle ACB (Prop. 21),

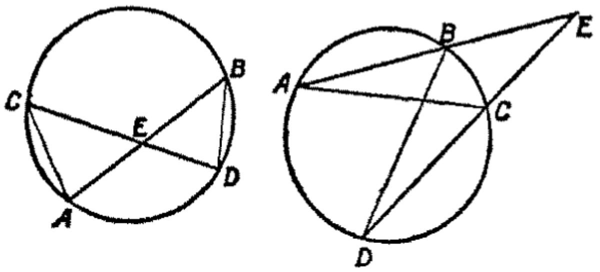
$\therefore\ \angle APB + \angle AQB = \frac{1}{2}$ (sum of the convex and concave angles)

$= \frac{1}{2}$ (four rt. $\angle$s)

$= 2$ rt.$\angle$s.

PROPOSITION 23.

In any circle the product of the segments made by the intersection of two chords with the circumference are equal.



Let AB, CD be any two chords which intersect either within or without the circle at E.

Join AC, BD. Then $\angle AEC = \angle BED$,

and $\angle ACE = \angle DBE$. (Prop. 21, Cor. 1.)

Therefore the $\triangle$s ACE, BDE are equiangular (Prop. 4),

$$\text{and } \therefore \frac{AE}{CE} = \frac{DE}{BE} \quad \text{(Prop. 13).}$$

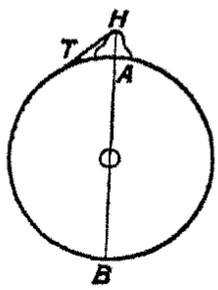
or $AE \cdot BE = CE \cdot DE$.

Cor. 1.—Conversely, when the products of the segments are equal, the points A, B, C, D lie on a circle.

Cor. 2.—In any circle the square on the tangent is equal to the product of the segments cut off on the secant.

Cor. 3.—Tangents to a circle from the same point are equal.

Example.



To find the distance of the horizon at sea.

Let the observer be at H at an elevation AH above the sea-level.

Then, if HT be a tangent to the surface of the water,

$$HT^2 = HB \cdot HA \quad \text{(Prop. 23, Cor. 2).}$$

$\therefore$ the distance of the horizon HT

$$= \sqrt{HB \cdot HA}$$
$$= \sqrt{(d+h)h}\ ,$$

if d is the Earth's diameter, and h is the elevation of the observer above sea-level.

Since $HT^2 = (d + h)h$, where d is about 8000 miles, and h is usually measured in feet,

$$\text{we have } \overline{HT^2} = \left(8000 + \frac{h}{5280}\right)\frac{h}{5280} \text{ in miles.}$$
$$= \frac{8000h}{5280}\ \text{, since } \frac{h^2}{5280^2}\ \text{, being very small,}$$

may be neglected without sensible error.

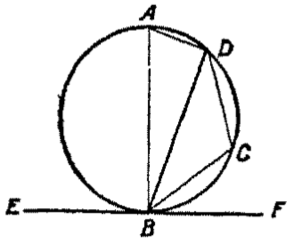
$$\therefore \overline{HT^2} = \frac{100}{66}h = \frac{3}{2}h \text{ approximately.}$$

Hence the rule:

Three times the height of the observer above the sea level in feet is equal to twice the square of the distance seen in miles.

PROPOSITION 24.

If a straight line touch a circle, and from the point of contact another straight line be drawn cutting the circle, the angles which this straight line makes with the first at the point of contact are equal to the angles in the adjacent segments.



If EF be a tangent, and BD a secant at the point B, Then $\angle DBF = \angle BAD$, and $\angle DBE = \angle BCD$.

Proof.—Draw BA perpr. to EF at the point B,

Then $\angle ADB$ is a rt. $\angle$. (Prop. 21, Cor. 2.)

and $\therefore \angle$s ABC, BAD = a rt. $\angle$ (Prop. 4.)

$\therefore \angle ABF = \angle ABD + \angle BAD$.

Take away the common $\angle ABD$.

Then the remaining angle DBF = the remaining angle BAD,
i.e., the $\angle DBF$ is equal to the angle in the adjacent segment BAD.

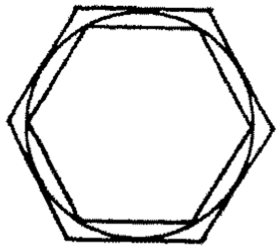
$$\text{Again,} \because \text{ the } \angle\text{s } BAD, BCD = 2 \text{ rt. } \angle\text{s} \quad \text{(Prop. 22)}$$
$$\text{and the } \angle\text{s } DBF, DBE = 2 \text{ rt. } \angle\text{s}$$
$$\therefore \angle\text{s } BAD, BCD = \angle\text{s } DBF, DBE.$$
$$\text{But } \angle BAD = \angle DBF \quad \text{(By the preceding part),}$$
$$\therefore \angle BCD = \angle DBE.$$

Exercises on the Circle.

1. To draw a triangle, having given the base, the vertical angle, and the altitude.
2. To draw a rt.- $\angle$d $\triangle$ when the hypotenuse and one side are given.
3. To draw a tangent to a circle from a given point without it.
4. To inscribe an equilateral $\triangle$ and a regular hexagon in a circle.
5. To inscribe a square and a regular octagon in a circle.
6. To find a mean proportional between two given straight lines.
7. Divide a circle into two segments, so that the angle contained in the one may be three times the angle contained in the other.
8. If a quadrilateral figure be described about a circle, the sums of the opposite sides will be equal to one another.
9. DF is a tangent to a circle, and terminated at D and F by two tangents drawn at the extremities of a diameter AB; show that the segment DF subtends a rt. $\angle$ at the centre of the circle.
10. If a circle be inscribed, in a rt.- $\angle$d $\triangle$, the excess of the two sides over the hypotenuse is equal to the diameter of the circle.
11. Two circles touch one another in A, and have a common tangent BC. Show that the angle BAC is a rt. $\angle$.

PROPOSITION 25.

To find the ratio of the circumference of a circle to its diameter.



If two similar polygons be drawn, one inscribed in a circle, and the other circumscribed about it, it is evident that the circumference of the circle is greater than the first and less than the second. The following table gives the lengths of the perimeters of regular polygons inscribed said circumscribed to a circle.

Number of Sides.	Perimeter of Inscribed Polygon. $d\times$	Perimeter of Circumscribed Polygon. $d\times$
6	3.00000	3.46410
12	3.10583	3.21539
24	3.13263	3.15966
48	3.13935	3.14609
96	3.14103	3.14271
192	3.14145	3.14187
384	3.14156	3.14166
768	3.14158	3.14161
1536	3.14159	3.14160
3072	3.14159	3.14159

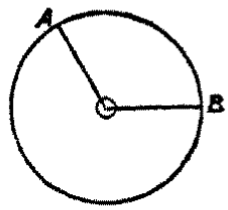
It is evident, therefore, that the circumference of a circle is equal to 3.14159 times the diameter to five places of decimals. The exact value is an incommensurable number, that is to say, it cannot be expressed exactly in figures. The Greek letter π (pi) is for convenience used to denote the *true* value, and the circumference of a circle is therefore π times its diameter. For all ordinary work 3.1416 is sufficiently accurate, and the vulgar fraction $\frac{355}{113}$ also expresses the value of π correctly to six decimal places. For rough calculations $\frac{22}{7}$ is frequently used, which is nearly correct to three decimal places.

PROPOSITION 26.

To find the length of a circular arc.

To find the length of an arc of a circle of given radius r, subtending a given angle of n degrees.

Since the arcs of a circle are in proportion to the angles which they subtend,

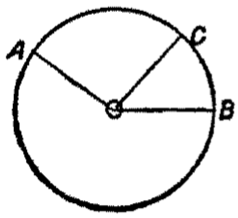


$$\frac{arcAB}{circumference} = \frac{\angle AOB}{4 \text{ rt. } \angle \text{s}}$$

$$\text{whence arc } AB = 2\pi r \times \frac{n^\circ}{360^\circ} = \frac{\pi r n^\circ}{180^\circ}$$

Measurement of angles in radian measure.—In mathematical work it is frequently convenient and necessary to measure angles without reference to any arbitrary unit, such as a degree or grade, by the ratio of the arc subtending it to the radius, on any circle.

Thus, the angle AOB is evidently determined if the ratio of the arc AB to the radius OB is known.



If, then, angles are measured by the ratio arc/radius, the unit angle will be that for which the ratio arc/radius = 1, or for which arc = radius.

This angle is called a *radian*. If arc BC = radius, then $\angle COB$ is a radian, and the length of any arc AB = radius $\times$ number of radians in $\angle AOB$.

$$\text{For an angle of } 180^\circ, \frac{\text{arc}}{\text{radius}} = \frac{\pi r}{r} = \pi \text{ units,}$$

$$\textit{i.e.,}\ \pi \text{ radians} = 2 \text{ rt. } \angle\text{s.}$$

$$\text{Hence 1 radian } = \frac{180^\circ}{\pi} = \frac{180^\circ}{3.14159} = 57^\circ 2958 \text{ nearly.}$$

Exercises.

1. Turn into radians 60°, 90°, 300°, 52°30′, 135°4′57″.

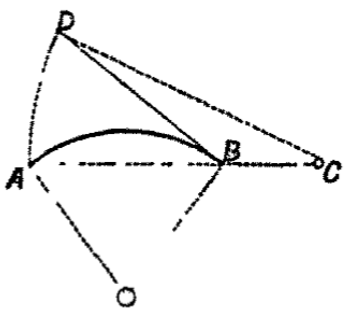
ANSWER: $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{5\pi}{3}$, $\frac{7\pi}{24}$, .75046 π.

2. Turn into degrees $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2}{3}$, $\frac{7}{15}$ radians.

ANSWER: 60°, $25\frac{5°}{7}$, $\frac{120°}{\pi}$, $\frac{84°}{\pi}$.

3. An arc of a circle whose radius is 6″, subtends an angle of 2 radians; how many degrees will be subtended by the same arc in a circle of 4″ radius?

ANSWER: $\frac{540}{\pi}$ degrees.



Graphical determination of the length of a circular arc.

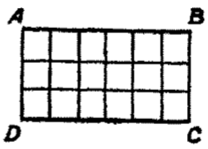
Let AB be the circular arc whose length is required. Join AB, and produce it backward, making $BC = \frac{1}{2}AB$. With C as centre, and radius CA, describe an arc of a circle cutting the tangent at B in the point D. Then BD is approximately equal to the arc AB.

If $\angle AOB$ exceed 45°, it is best to treat the arc in two parts.

Numerical Value of Areas.

PROPOSITION 27.

To find the area of a rectangle.



The unit of area is a square, whose sides are equal to the unit of length. Thus, if 1 inch is the linear unit, 1 sq. inch is the unit of area.

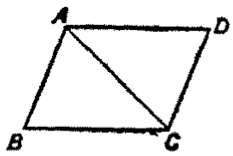
Let $ABCD$ be a rectangle, and let there be $l$units in the length AB, and b units in the breadth AD. Then, if AB be divided into l

equal parts, and lines be drawn through the points parallel to AD, the whole rectangle is divided into l new rectangles of unit width.

If now AD be divided into b equal parts, and lines be drawn parallel to AD, each of the former rectangles is divided into b new rectangles, whose sides are of unit length. Therefore, the whole number of units of area is $b \times l$, the product of the linear dimensions of the length and breadth.

PROPOSITION 28.

To find the area of a parallelogram.

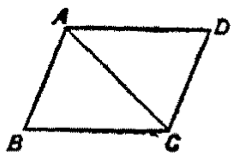


Let $ABCD$ be a ▱. Then (by Prop. 9), the area of the ▱ is equal to the area of a rectangle, $A'BCD'$, having the same base and altitude.

∴ the area of ▱ = base × the altitude.

PROPOSITION 29.

To find the area of a triangle.



Let ABC be a triangle.

Draw CD parallel to AB and AD parallel to M.

Then $ABCD$ is a parallelogram.

∴ area = base × altitute.

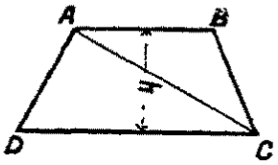
But $\triangle ABC = \frac{1}{2}$ ▱$ABCD$ (by Prop. 8).

∴ area of a $\triangle = \frac{1}{2}$ base × the altitude.

PROPOSITION 30.

To find the area of a trapezium.

Let $ABCD$ be a trapezium. Join AC.



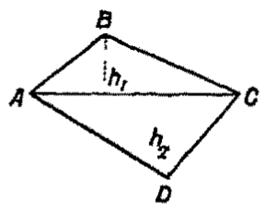
Then area of $\triangle ADC = \frac{1}{2}\,DC \times h$,
and area of $\triangle ABC = \frac{1}{2}\,AB \times h$.

$$\therefore \text{ the area of the trapezium } = \frac{1}{2}\,DC \times h + \frac{1}{2}\,AB \times h$$
$$= \frac{1}{2}\,h(AB + DC).$$

I.e., the area of a trapezium = the altitude × $\frac{1}{2}$ the sum of the parallel sides.

PROPOSITION 31.

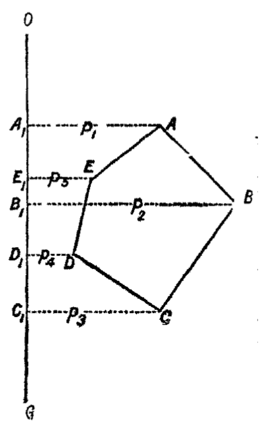
To find the area of a quadrilateral.



Let $ABCD$ be a quadrilateral. Join AC.
Then area of $\triangle ABC = \frac{1}{2}\,AC \times h_1$,
and area of $\triangle ACD = \frac{1}{2}\,AC \times h_2$,
$\therefore$ area of the quadrilateral
$= \frac{1}{2}\,AC \times h_1 + \frac{1}{2}\,AC \times h_2$
$= \frac{1}{2}\,AC(h_1 + h_2)$

I.e., the area of a quadrilateral = $\frac{1}{2}$ product of a diagonal into the sum of the perpendiculars on the diagonal from the opposite vertices.

PROPOSITION 32.



To find the area of a polygon.

Let $ABCDE$ be any polygon.

Draw any line OG, and drop perps. upon it from the angles of the polygon, *viz.*, p_1, p_2, p_3, p_4, p_5; and let a_1, a_2, a_3, a_4, a_5, be the distances of these perpendiculars from any point O in the line OG.

Then the area of the polygon

$$\begin{aligned} ABCDE &= \text{trap. } AA_1B_1B + \text{trap. } BB_1C_1C \\ &\quad - \text{trapms } AA_1E_1E, EE_1D_1D, DD_1C_1C \\ &= \frac{1}{2}(p_1+p_2)(a_2-a_1) + \frac{1}{2}(p_2+p_3)(a_3-a_2) \\ &\quad - \frac{1}{2}(p_1+p_5)(a_5-a_1) - \frac{1}{2}(p_5+p_4)(a_4-a_5) \\ &\quad - \frac{1}{2}(p_4+p_3)(a_3-a_4)\,, \end{aligned}$$

which reduces to

$$\frac{1}{2}\{p_1a_2 - p_2a_1 + p_2a_3 - p_3a_2 + p_3a_4 \\ - p_4a_3 + p_4a_5 - p_5a_4 + p_5a_1 - p_1a_5\}$$

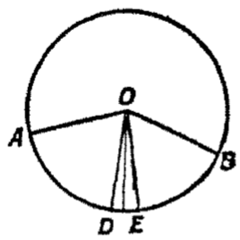
an expression which can be easily written down, on account of its symmetry.

This method is convenient in taking out areas in surveys, &c.

PROPOSITION 33.

To find the area of a circle and its sector.

Let AOB be a sector of a circle.



Then if AOB be supposed divided into n smaller equal sectors, such as ODE, and DE be joined, the area of the $\triangle ODE$ approximates more nearly to the area of the sector ODE, as the number of $\triangle$s is increased, and when, therefore, the number is indefinitely great, the error is indefinitely small.

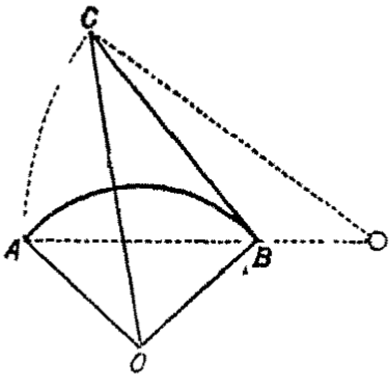
$$\begin{aligned} \text{But the area of each } \triangle &= \frac{1}{2} \text{ base } \times \text{ the altitude,} \\ \therefore \text{ the total area} &= \frac{1}{2} \text{ (sum of the bases) } \times \text{ the altitude} \\ &= \frac{1}{2} \text{ arc } AB \times \text{ the radius,} \end{aligned}$$

when the number of $\triangle$s is infinitely great.

$$\therefore \text{ the area of a sector} = \frac{1}{2} \text{ length of its arc } \times \text{ the radius.}$$

In the case of a complete circle, the arc = the circumference = $2\pi r$,
and $\therefore$ the area of a circle = $\frac{1}{2} \times 2\pi r \times r = \pi r^2$

$$= \frac{\pi d^2}{4} = 0.7854 d^2$$



To find graphically the area of a circular sector.

Let OAB be a sector. Draw BC equal in length to the arc AB. (Prop. 26.)

Join OC. Then $\triangle OBC$ = sector OAB.

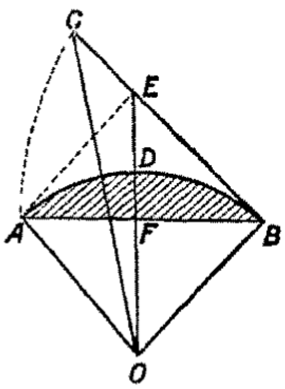
Proof.—For area of the sector = $\frac{1}{2}$ arc $\times$ the radius, and the area of the $\triangle$ = $\frac{1}{2}$ $OB \times BC$, but OB = radius, and BC = the length of the arc.

$\therefore$ area of the $\triangle$ $OBC = \frac{1}{2}$ $arc \times$ the radius,
and $\therefore$ area of the sector OAB = area of $\triangle$ OBC.

PROPOSITION 34.

To find the area of a circular segment.

Let ADB be the circular segment.



Then

$$\text{area} = \text{area of the sector } OADB - \triangle OAB$$
$$= \frac{1}{2}\, arc \times r - \frac{1}{2}\, AB \times OF.$$

Graphically.—By Prop. 33 construct a $\triangle OBC$ = sector $OADB$.

Draw AE parallel to OB, and join OE.

Then $\triangle OEB = \triangle OAB$ (Prop. 9, Cor. 2),

$$\therefore \text{ the area of the segment} = \triangle OBC - \triangle OEB$$
$$= \triangle OEC.$$

PROPOSITION 35.

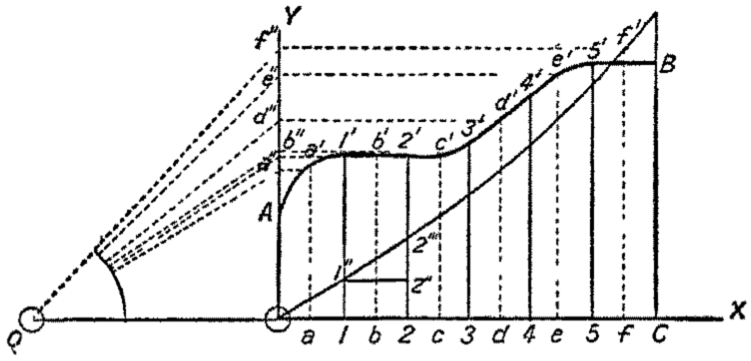
To find areas by a sum-curve.

Let $OABC$ be an area bounded by the axes OX, OY.

Divide the base into a number of parts (preferably equal), *viz.*, 1, 2, 3, &c., and draw the vertical ordinates $11'$, $22'$, &c. Also draw the mid-ordinates aa', bb', cc', &c., and project them on the axis OY at the points a'', b'', c'', &c.

Take any pole Q in OC produced, and join Qa'', Qb'', Qc'', &c.

Beginning at O, draw $O1'' \parallel Q''$, $1''2'' \parallel Qb''$, and so on.



Then the area under the curve $OABC$ up to any point is given approximately by the product of the ordinate of the curve $O1''2''$, &c., up to that point into the polar distance OQ. This curve is called a *sum-curve*.

Proof.—For consider any $\triangle 1''2''2'''$ and the $\triangle QOb''$. Since these $\triangle$s have their sides respectively parallel (by construction) they are similar,

$$\text{and } \therefore \frac{2''2'''}{1''2''} = \frac{Ob''}{QO}$$

and $\therefore$ $2''2''' \times QO = Ob'' \times 1''2'' = bb' \times 12 = \text{area } 11'2'2$ approximately.

I.e., the area $11'2'2$ which has been added on in going from 1 to 2 is equal to the increase in the ordinate of the Sum-Curve $\times$ the polar distance, and similarly for the other elements of area. Wherefore, if we start from O, the area up to any point is equal to the ordinate up to that point $\times$ the polar distance.

Solid Geometry.

42. A *polyhedron* is a figure bounded on all sides by planes.

43. A *prism* is a polyhedron whose sides are parallelograms, and whose extremities are equal polygons in parallel planes.

44. A *parallelopiped* is a polyhedron bounded by three pairs of parallel planes.

45. A *pyramid* is a polyhedron, one of whose faces is a polygon, and the others triangles, whose bases are the sides of the polygon, and having a common vertex.

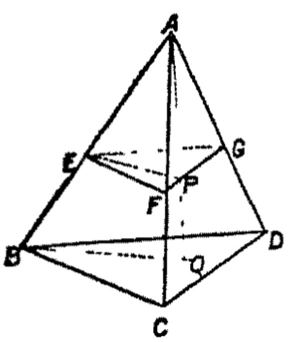
46. A *tetrahedron* is a pyramid on a triangular base.

47. A *frustum* of a solid is that portion of it contained between the base and another plane which cuts the solid.

48. A *prismoid* is a solid whose ends are similar figures, having their sides parallel, and in parallel planes, or it is a frustum of a pyramid.

PROPOSITION 36.

The areas of the sections of a pyramid made by planes parallel to the base, are proportional to the squares of their distances from the vertex.



Let $ABCD$ be a pyramid on a $\triangle$r base BCD, and let EFG be a parallel section.

Draw APQ perpendicular to the base, meeting the parallel planes in P and Q. Join EP, BQ.

Then $\because EF \parallel BC, EG \parallel BD$, $FG \parallel CD$, the $\triangle EFG$ is equiangular to the $\triangle BCD$, and the $\triangle AEF$ is similar to the $\triangle ABC$, and the $\triangle AEP$ to the $\triangle ABQ$,

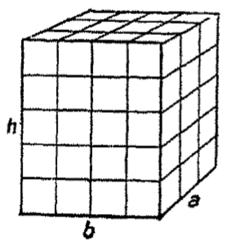
$$\therefore \frac{\text{area } EFG}{\text{area } BCD} = \frac{\overline{EF}^2}{\overline{BC}^2} = \frac{\overline{AE}^2}{\overline{AB}^2} = \frac{\overline{AP}^2}{\overline{AQ}^2} \quad \text{(Prop. 16 and 13.)}$$

Also, if the pyramid is on a polygonal base, it can be decomposed into pyramids on $\triangle$r bases, and the theorem proved in the same manner.

COR.—If two pyramids are on equal bases, and have equal altitudes, the sections of them at equal distances from the base are equal.

PROPOSITION 37.

The volume of a right prism is equal to the area of its base multiplied by the height.

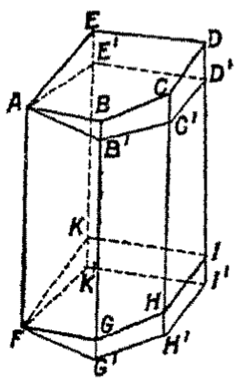


For, if a, b be the length and breadth of the base, there are $a \times b$ units of area in the base; and on each unit of area in the base there are as many units of volume as there are units of length in the height h.

$\therefore$ the whole volume $= abh$.

PROPOSITION 38.

The volume of an oblique prism is equal to the area of its right section multiplied by its length.



Let $ABCDEFGHIK$ be an oblique prism, and let a section at right $\angle$s to its edges be made by the plane $AB'C'D'E'$. Then, if the wedge-shaped solid $ABCDEB'C'D'E'$ be placed at the other extremity, A falling on F, B on G, and so on, the volume of the original prism will be equal to the volume of the right prism

$AB'C'D'E'FG'H'I'K'$ = area of its base $\times$ height (Prop. 37).

COR. 1.—The volume of a parallelopiped = the products of its base into its altitude.

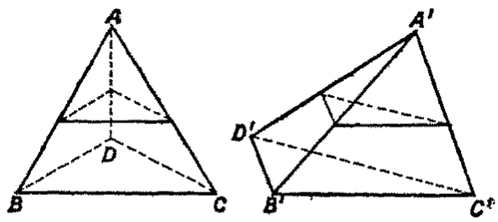
COR. 2.—The volume of an oblique $\triangle$r prism = area of its base multiplied by the altitude; for it is half the volume of a parallelepiped.

COR. 3.—The volume of any oblique prism = area of its base $\times$ the altitude, for it may be divided up into triangular prisms.

PROPOSITION 39.

Pyramids on equal bases and of equal altitude are equal in volume.

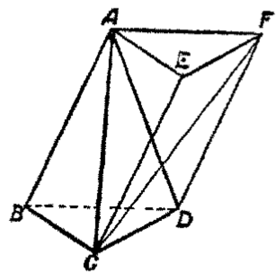
Let $ABCD$, $A'B'C'D'$ be pyramids on equal bases and having the same altitude.



Then, if each be supposed divided into the same number of infinitely thin layers, since the corresponding layers in each are equal in area (Prop. 36, Cor.) and their thickness is the same, the sum of all those in the first must be equal to the sum of all those in the second, and the volumes of the pyramids are therefore equal.

PROPOSITION 40.

The volume of a pyramid is one-third of the prism standing on the same base.



Let $ABCD$ be a pyramid. Through C, D draw lines parallel to AB, and cut them by a plane AEF parallel to BCD, forming a prism $ABCDEF$. Join CF. Then the prism is divided into three pyramids $ABCD$, $CEAF$, and $FDCA$.

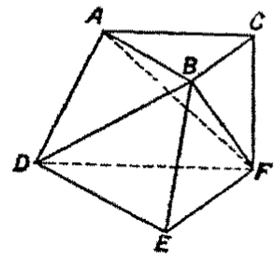
But $ABCD = CEAF$, being on equal bases BCD, AEF, and having the same altitude (Prop. 39). Also $ABCD = FDCA$, being on equal bases ABD, FAD, and having the same altitude (Prop. 39).

$$\therefore \text{each pyramid} = \frac{1}{3} \text{ prism}.$$

Cor. 1.—The volume of a $\triangle$r pyramid = $\frac{1}{3}$ base $\times$ altitude.

Cor. 2.—The volume of any pyramid = $\frac{1}{3}$ base $\times$ altitude.

PROPOSITION 41.



To find the volume of a frustum of a triangular pyramid between parallel planes, in terms of its altitude and the areas of its bases.

Let $ABCDEF$ be a frustum of a pyramid, the plane ABC being $\parallel$ DEF.

Let it be cut by planes BDF, BAF into three pyramids, viz., $BDEF$, $FABC$, $BADF$.

Let the areas of ABC, DEF be b, B, and the altitude of the frustum be h. Then $BDEF = \frac{1}{3}\, h \cdot B$ (Prop. 40, Cor. 1) and $FABC = \frac{1}{3}\, h \cdot b$ (Prop. 40, Cor. 1).

$$Also\ \frac{BADF}{BACF} = \frac{\text{base } ADF}{\text{base } ACF} = \frac{DF}{AC} \qquad \text{(Prop. 11, Cor.)} = \frac{\sqrt{B}}{\sqrt{b}} \quad \text{(Prop. 16.)}$$

$$\text{and } \therefore BADF = \frac{\sqrt{B}}{\sqrt{b}} \cdot BACF = \frac{\sqrt{B}}{\sqrt{b}} \frac{1}{3}\, bh = \frac{1}{3}\, h\sqrt{Bb}$$

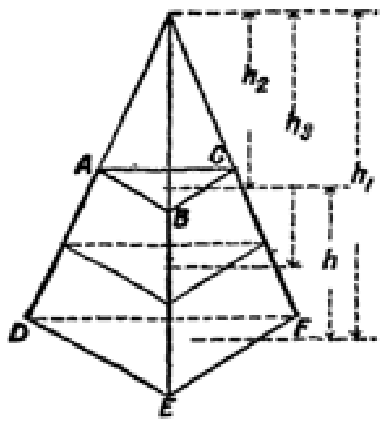
$$\therefore \text{the volume of the frustum} = \frac{1}{3}\, h\{B + \sqrt{Bb} + b\}$$

that is, is equal to the volume of three pyramids having the same height as the frustum, and having bases respectively equal to the parallel faces of the frustum, and the geometric mean between them.

COR.—This property may be extended to the frusta of all pyramids between parallel planes, by considering them as made up of the frusta of pyramids on triangular bases.

Trapezoidal Formula.

The formula $\frac{1}{3}\ h\left\{B+\sqrt{Bb}+b\right\}$ may be written in another form which is convenient in the calculation of earthwork.



Let $h_1 h_2 h_3$ be the distances of the bottom, top and middle of the frustum from the vertex.

Then if M be the area of the middle plane,

$$\frac{M}{b}=\left(\frac{h_3}{h_2}\right)^2 \qquad \text{but } h_3=\frac{h_1+h_2}{2}$$

$$\therefore M=b\left(\frac{h_1+h_2}{2h_2}\right)^2=\frac{1}{4}\,b\left(\frac{h_1^2+h_2^2+2h_1h_2}{h_2^2}\right)$$

$$\text{but } \frac{h_1^2}{h_2^2}=\frac{B}{b} \qquad \therefore \frac{h_1^2+h_2^2}{h_2^2}=\frac{B+b}{b}$$

$$\therefore M = \frac{1}{4} b \left\{\frac{B+b}{b} + 2\frac{h_1}{h_2}\right\} = \frac{1}{4} b \left\{\frac{B+b}{b} + 2\sqrt{\frac{B}{b}}\right\}$$
$$= \frac{B+b}{4} + \frac{1}{2}\sqrt{Bb}$$

$\therefore$ 4 times the middle area $= B + b + 2\sqrt{Bb}$

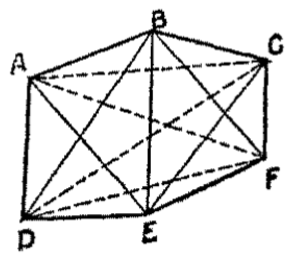
Now, $\frac{1}{3} h \left\{B + \sqrt{Bb} + b\right\} =$

$$\frac{h}{6} \left\{B + b + B + b + 2\sqrt{Bb}\right\}$$

= (Sum of the end areas + 4 times the middle area) multiplied into one-sixth the height of the frustum.

PROPOSITION 42.

To find the volume of the frustum of a triangular prism.



Let $ABCDEF$ be the frustum.

Let the base $DEF = b$, and let h, h_1, h_2 be the altitudes of A, B, C above the plane DEF.

First cut the frustum by a plane BDF.

The volume of the pyramid $BDEF = \frac{1}{3}\ bh$. (Prop. 40, Cor. 1.)

Divide the remainder by a plane BAF into pyramids $BADF$, $ABCF$.

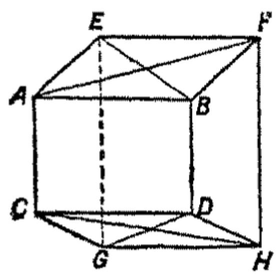
Now since $BADF$, $EADF$ stand upon the same base ADF and have equal altitudes (for $BE \parallel AD$), $BADF = EADF = \frac{1}{3}\ bh$ (Prop. 40, Cor. 1);
also $ABCF = DBCF = DECF$,
$\because$ they stand upon equal bases and have the same altitude,
$\therefore ABCF = \frac{1}{3}\ bh_2$.

$\therefore$ the volume of the whole frustum

$$= BDEF + BADF + BACF$$
$$= \frac{1}{3}\ bh_1 + \frac{1}{3}\ bh + \frac{1}{3}\ bh_2.$$
$$= \frac{1}{3}\ b(h_1 + h_2 + h).$$

COR. 1.—In a right prism the volume of the frustum = the area of its right section multiplied by onethird the sum of the parallel edges.

COR. 2.—The volume of the frustum of an oblique prism = the area of its right section $\times \frac{1}{3}$ sum of the parallel edges.



A useful corollary to Prop. 42 is the case of a prism bounded by plane surfaces at the ends, which are parallelograms or regular polygons.

Let the figure represent such a solid, where the ends $AEFB$, $CDHG$ are ▱s or regular polygons, and let h_1, h_2, h_3, h_4 be the altitudes of the points A, B, F, E above the plane $CDHG$, respectively.

Then the volume of the triangular prisms $ABFHDC$ is $\frac{1}{3}\ \overline{CDH}(h_1+h_2+h_3)$.

Also $AEFHGC$ is $\frac{1}{3}\ \overline{CGH}(h_1 + h_4 + h_3)$.

But if the ends are ▱s or regular polygons the diagonal CH divides the base into equal areas, each equal to $\frac{B}{2}$ where B is the area of the base

$\therefore$ the volume of the whole solid is $= \frac{1}{3} \cdot \frac{B}{2}(2h_1 + 2h_3 + h_2 + h_4)$.

Similarly, if the diagonals BE, DG be drawn, it may be shown that the total volume is $\frac{1}{3} \cdot \frac{B}{2}(2h_4 + 2h_2 + h_1 + h_3)$

$$
\begin{aligned}
\therefore 2 \text{ volume} &= \frac{B}{6}\ (3h_1 + 3h_2 + 3h_3 + 3h_4) \\
&= \frac{B}{2}\ (h_1 + h_2 + h_3 + h_4) \\
\therefore \text{the volume } V &= \frac{1}{4}\ B(h_1 + h_2 + h_3 + h_4)
\end{aligned}
$$

and similarly, for any other regular polygon of n sides

$$V = \frac{1}{n} \cdot B(h_1 + h_2 + h_3 + \cdots + h_n)$$

In calculating the earthwork taken from borrow pits, the ground is staked out in squares, which must be small enough that their surfaces may be considered planes, without sensible error. The height of the horizontal bottom of the pit being taken with a level, and the heights of the corners of the square above a known datum being already determined, the differences give the heights of the corners, and the volumes can then be determined as above.

In laying out the squares, the tape must be held horizontally.

49. A *wedge* is a polyhedron whose base is a trapezium and whose edge is parallel to the base.

PROPOSITION 43.

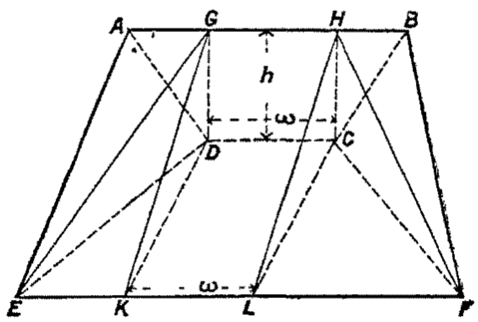
To find the volume of a wedge.

Let $ABCDEF$ be a wedge.

Then the volume $KLCHGD$ = base $CHL\ \times$ altitude CD.

$$= \frac{1}{2}\, hl \times w,$$

where l is the altitude of the wedge. (Prop. 37.)



Also the volume of the pyramid $EDGA$

$$= \frac{1}{3} \text{ base } ADG \times \text{ height.} \quad \text{(Prop. 40, Cor. 1.)}$$

$$= \frac{1}{3} \cdot \frac{1}{2} \cdot AG \cdot hl,$$

and the volume of the pyramid $FCHB$

$$= \frac{1}{3} \text{ base } CHB \times \text{ height}$$

$$= \frac{1}{3} \cdot \frac{1}{2} \cdot BH \cdot hl \quad \text{(Prop. 40, Cor. 1).}$$

Also the volume of the pyramid *GDKE*

$$= \frac{1}{3}\,\mathrm{GDK} \times EK = \frac{1}{3} \cdot \frac{1}{3}\, GD \times l \times EK,$$

and the volume of the pyramid HCLF

$$= \frac{1}{3}\,\text{base}\; HCL \times LF = \frac{1}{3} \cdot \frac{1}{2}\, \overline{HC} \cdot l \times \overline{LF}$$

$$\begin{aligned}
\therefore \text{ the whole volume} &= \frac{1}{2}\, whl + \frac{1}{6}\, \overline{AG} \cdot hl + \frac{1}{6}\, \overline{BH} \cdot hl + \frac{1}{6}\, hl \cdot \overline{EK} + \frac{1}{6}\, hl \cdot \overline{LF}. \\
&= \frac{lh}{6}\, \{3w + AG + BH + EK + LF\} \\
&= \frac{lh}{6}\, \{(w + EK + LF) + w + (w + AG + BH)\} \\
&= \frac{lh}{6}\, (EF + w + AB)\,, \text{ that is,}
\end{aligned}$$

Add to the edge of the wedge the sums of the parallel sides of the base, and multiply the result by onesixth of the width of the base multiplied by the altitude of the wedge.

VOLUMES BOUNDED BY CURVED SURFACES.

The Cylinder.

50. A *cylinder* is a solid generated by a line which moves always parallel to itself.

51. A *right circular cylinder* is the solid generated by the revolution of a rectangle about one of its sides.

PROPOSITION 44.

To find the lateral surface and volume of a right circular cylinder.

Since the lateral surface = the area of a rectangle, whose base is equal to the circumference of the cylinder = πd, and whose height is equal to its height

h, the lateral surface $= \pi dh$, where d is the diameter of the cylinder and h its height.

Again, since the cylinder may be regarded as a right prism, whose base is a polygon having an infinite number of sides,
its volume $=$ area of the base $\times$ the length $= \pi r^2 h$, r being the radius of the base.

The Cone.

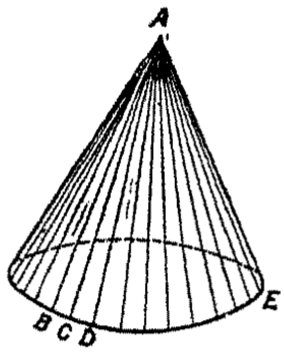
52. A *cone* is a solid generated by the movement of a straight line which always passes through a fixed point.

53. A *right circular cone* is the solid generated by the revolution of a right-angled triangle about one of the sides containing the right angle.

PROPOSITION 45.

To find the lateral surface and volume of a right circular cone.

Let $ABCDE$ be a right circular cone.



Inscribe in its base a regular polygon BCD ... , and let planes ACD, &c., be drawn. Thus a polygonal pyramid is inscribed within the cone, and when the sides of the polygon are infinite in number the lateral surface and volume of the pyramid are equal to the lateral surface and volume of the cone.

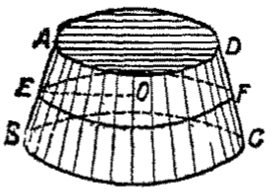
But the area of a triangle $ACD = \frac{1}{2}\,CD \times$ the perpendicular from A on CD, and $\therefore$ the lateral surface of the pyramid $= \frac{1}{2}$ perimeter of the polygon $\times$ the perpendicular from A on one of the sides, and in the limit, the circumference of the polygon = circumference of the ○ and the perpendicular from A on the side $= AC$. $\therefore$ the lateral surface of the cone $= \frac{1}{2}$ circumference of the base $\times$ slant side.

Again, since volume of a pyramid$= \frac{1}{3}$ base $\times$ height, the volume of a cone = $\frac{1}{3}$ base $\times$ height = $\frac{1}{3}\pi r^2 h$.

PROPOSITION 46.

To find the lateral surface of the frustum of a cone.

Let $ABCD$ be the frustum, and let EF be midway between BC and AD. Then the surface may be regarded as made up of an infinite number of trapezia, whose parallel sides are the sides of regular polygons inscribed in the circular ends. But area of a trapezium = $\frac{1}{2}$ altitude $\times$ sum of the parallel sides, and $\therefore$ sum of all the trapeziums = $\frac{1}{2}$ altitude $\times$ sum of the circumferences of the parallel ends.



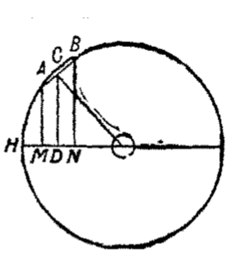
$\therefore$ the lateral surface of the frustum slant = $\frac{1}{2}$ slant side $\times$ sum of the circumferences of the parallel ends = $\frac{1}{2}$ slant side $\times$ circumference of the circle midway between the ends = $2\pi \cdot AB \cdot EO$.

The Sphere.

54. A *sphere* is the solid generated by the revolution of a semicircle about its diameter.

PROPOSITION 47.

To find the surface of a sphere.



The zone of the sphere generated by a small arc AB is ultimately equal to the surface of the frustum of a cone, whose slant side is AB and axis HO, when the chord AD becomes infinitely small; and the surface of the sphere is the sum of all such zones. Bisect AB in C, and draw CD perpendicular to HO.

The surface of the frustum = $2\pi \cdot CD \cdot AB$ (Prop. 46).

Join CO, and let MN be the projection of AB on HO: then by similar triangles $\frac{CD}{CO} = \frac{MN}{AB}$

and $\therefore CD \cdot AB = CO \times MN$

$\therefore$ the surface of the frustum $= 2\pi CO \times MN$

and $\therefore$ when the zone is infinitely narrow $= 2\pi \cdot r \cdot MN$ Also the sum of all the projections of the arcs, such as $MN = 2r$

$$\therefore \text{surface of the sphere} = 2\ \pi\ r \times 2\ r = 4\pi r^2$$

COR. 1.—The area of any zone is in proportion to its height.

PROPOSITION 48.

To find the volume of a sphere.

Suppose the sphere to be made up of an infinite number of cones.

Then the volume of each cone $= \frac{1}{3}$ base $\times$ altitude $= \frac{1}{3}$ base $\times\ r$, where r is the radius of the sphere.

Also the whole volume of the sphere = sum of the volumes of all the cones

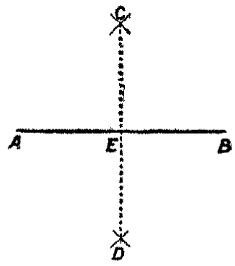
$$= \frac{1}{3}\, r \times \text{sum of all the bases}$$
$$= \frac{1}{3}\, r \times 4\pi r^2 = \frac{4}{3}\pi r^3 \quad \text{(Prop. 47).}$$

or since $r^3 = \frac{d^3}{8}$, the volume of the sphere $= \frac{\pi d^3}{6}$

PROBLEMS IN PLANE GEOMETRY FOUND USEFUL IN DRAWING.

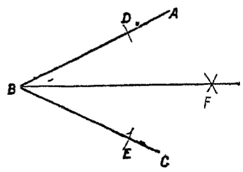
PROBLEM 1.—*To divide a straight line into two equal parts.*

Let AB be the straight line.



With A and B as centres, and with the same radius, describe arcs intersecting in C and D. Then a line from C to D bisects AB in the point E. (Proof by Prop. 5.)

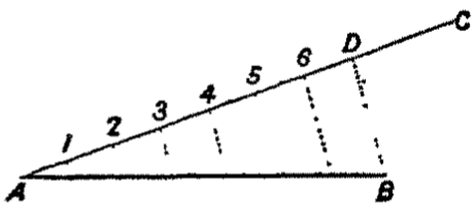
PROBLEM 2.—*To divide an angle into two equal parts.*



With B as centre describe an arc cutting BA and BC in D and E. With D and E as centres, and with the same or another radius describe arcs cutting in F.

Then BF bisects the angle ABC. (Proof by Prop. 7.)

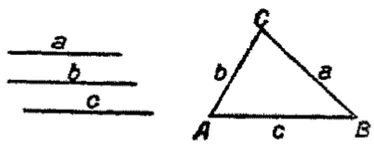
PROBLEM 3.—*To divide a line into any number of equal parts; seven, for example.*



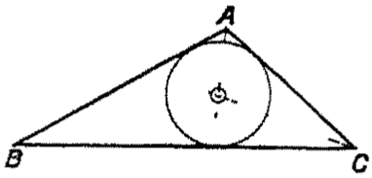
Set off a line AC, and on it mark off seven equal parts, starting from A, and ending at D. Join DB, and draw parallels through the points 1, 2, 3, &c. These will divide AB into seven equal parts. (Proof by Prop. 13)

PROBLEM 4.—*To draw a triangle, whose sides are of known length.*

Let a, b, c be the sides of the triangle. Take any one of the sides, such as c, and from its ends, draw arcs with radii equal to a and b intersecting in C. Join AC, BC. (Proof by Prop. 7.)



PROBLEM 5.—*To inscribe a circle in a given triangle.*

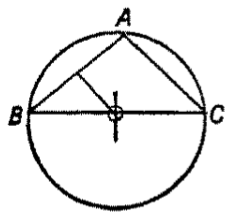


Let *ABC* be the triangle.

Bisect any two of the angles (Problem 2).

Then the point *O*, where the bisecting lines intersect, is the centre of the circle. (Proof by Prop. 5.)

PROBLEM 6.—*To circumscribe a circle about a given triangle.*



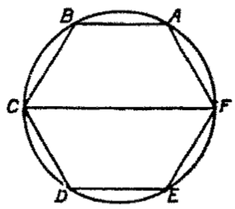
Let *ABC* be the triangle.

Bisect any two of the sides (Problem 1).

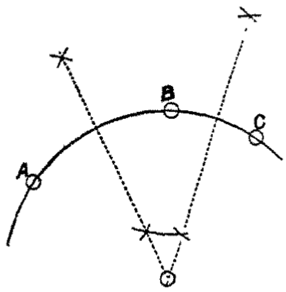
Then the point *O*, where the bisecting lines intersect, is the centre of the circle. (Proof by Prop. 6.)

PROBLEM 7.—*To inscribe a hexagon in a given circle.*

Set off the radius six times round the circle.



Then *ABCDEF* will be the hexagon required. (Proof by Prop. 6, Cor. 2.)



PROBLEM 8.—*To draw a circular arc through three given points.*

Let *A*, *B*, *C* be the points.

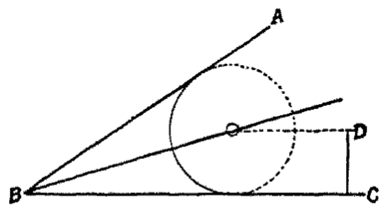
Bisect *AB* and *BC* (Problem 1.)

Then the point *O*, where the bisecting lines intersect, is the centre of the arc required. (Proof by Prop. 20, Cor. 1.)

PROBLEM 9.—*To inscribe in a given angle a circle of given radius.*

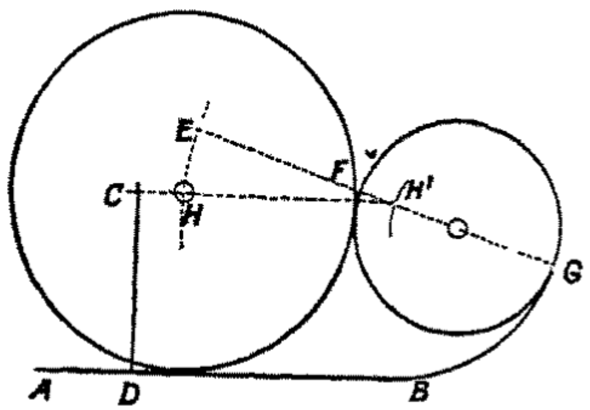
Let *ABC* be the given angle.

Bisect the angle (Problem 2).



Draw CD at right angles to BC and equal to the given radius. Draw DO parallel to BC. Then O is the centre of the circle required. (Proof by Prop. 5.)

PROBLEM 10.—*To describe a circle of given radius to touch a given line and a given circle.*



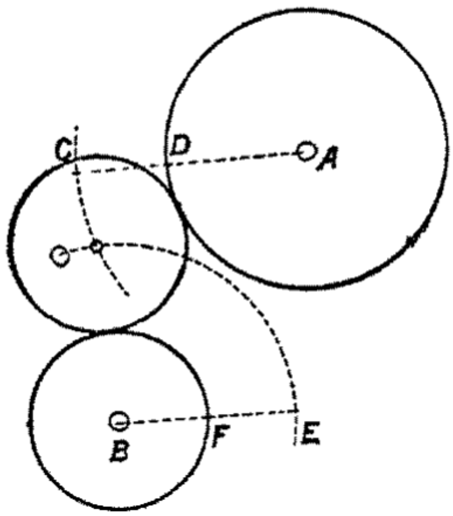
Draw any line OE, and make $FE =$ given radius.

Draw an arc EH about O. Set up $CD =$ radius, and draw $CH \parallel$ to AB. Then H is the centre of the circle required.

If it be required to touch on the other side at G, set off $GH' =$ given radius, and draw an arc cutting CH produced in H'. Then H' is the centre required.

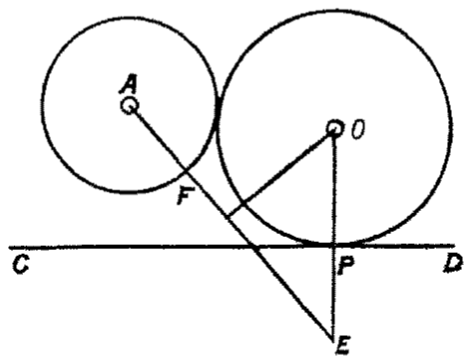
PROBLEM 11.—*To describe a circle, whose radius is given, to touch two given circles.*

Let A and B be the centres of the given circles.



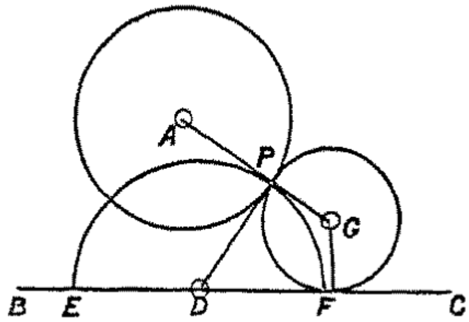
Set off CD, EF equal to the given radius, and draw arcs to intersect in O. Then O is the centre of the circle required.

PROBLEM 12.—*To describe a circle tangent to a given line at a given point, and touching a given circle.*



Let A be the centre of the given circle, CD the given line, and P the given point. Set off $PE = AF$ and bisect AE at right angles. Then O is centre of circle required.

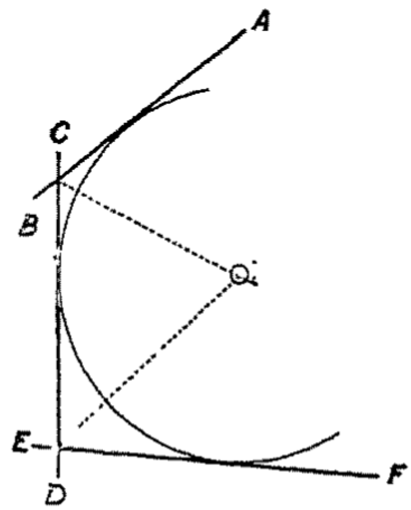
PROBLEM 13.—*To describe a circle tangent to a given line and touching a given circle in a given point.*



Let A be the centre of the given circle, P the point, and BC the given line. Join AP and produce. Draw PD perpendicular to AP. Describe semicircle EPF. Erect a perpendicular FG. Then G is centre of circle required.

PROBLEM 14.—*To draw a circle to touch three given straight lines.*

Let AB, CD, EF be the given lines.



Bisect the angles. The point of intersection of the bisectors will be the centre of the required circle.

This is equivalent to finding the centre of the escribed circle of a triangle.

BRADBURY, AGNEW, & CO. LD., PRINTERS, LONDON AND TONBRIDGE.

[The end of **Geometry for Technical Students**
by E. H. (Ernest Headly) Sprague]